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# ANTICANONICAL MODELS OF RULED SURFACES

Lucian Bădescu

Let  $X$  be a smooth projective surface over the complex field  $\mathbb{C}$ , and  $K$  a canonical divisor on  $X$ . Following Sakai [12] we denote by  $\phi_m$  the  $m$ -th anticanonical map defined by the linear system  $|-mK|$ , and by  $\kappa^{-1}(X)$  the anticanonical dimension of  $X$  defined as follows

$$\kappa^{-1}(X) = \begin{cases} \max_m \dim \phi_m(X), & \text{if } |-mK| \neq \emptyset \text{ for some } m > 0, \\ -\infty, & \text{if } |-mK| = \emptyset \text{ for every } m > 0. \end{cases}$$

We also define the  $m$ -th antigenus  $p_{-m}(X) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(-mK))$  and the anticanonical ring of  $X$ ,

$$R^{-1}(X) = \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(-mK))$$

In [12] Sakai studied in detail the rational surfaces  $X$  with  $\kappa^{-1}(X) = 2$  and gave many examples of such surfaces. His main result is the following:

Theorem 1 (Sakai [12]). Let  $X$  be a smooth projective rational surface with  $\kappa^{-1}(X) = 2$ . Then the anticanonical ring  $R^{-1}(X)$  is finitely generated. Denoting by  $Y = \text{Proj}(R^{-1}(X))$  (and calling it the anticanonical model of  $X$ ), then the following statements hold:

- i)  $Y$  has only rational singularities, and
- ii) Some positive multiple of an anticanonical divisor  $-K_Y$  of  $Y$  is an ample Cartier divisor.

Furthermore there is a canonical birational morphism  $u: X \rightarrow Y$ . If  $X$  contains no redundant exceptional curves (see [12], and also the definition below), then  $X$  coincides to the minimal desingularization of  $Y$ .

Conversely, if  $Y$  is a normal projective surface satisfying i) and ii), then the minimal desingularization  $X$  of  $Y$  is a rational surface with  $\kappa^{-1}(X) = 2$  and  $Y$  is isomorphic to  $\text{Proj}(R^{-1}(X))$ .

On the other hand, Sakai computed in [14] the anticanonical dimension of a geometrically ruled surface  $X = P(E) \rightarrow B$  over a smooth projective curve  $B$  of genus  $g \geq 0$ , with  $E$  a normalized (in the sense of [9], page 373) rank two vector bundle over  $B$ . In particular, he found that  $\kappa^{-1}(X) = 2$  iff  $E$  is of the form  $E =$

$= \mathcal{O}_B \otimes L^{-1}$ , with  $L$  a line bundle of degree  $e > 2g-2$ . If moreover the line bundle  $L^{2g-2} \otimes \omega_B^{-e}$  (of degree 0) is not a torsion, then Sakai also observed (loc. cit.) that  $R^{-1}(X)$  is not a finitely generated  $\mathbb{C}$ -algebra ( $\omega_B$  is of course the canonical line bundle of the curve  $B$ ).

In this paper we want to elucidate what is going on with the non-rational surfaces  $X$  over  $\mathbb{C}$  with  $K^{-1}(X) = 2$ , i.e. to find a "non-rational" counterpart of theorem 1.

Let then  $X$  be a smooth projective non-rational surface with  $K^{-1}(X) = 2$ . Then all the plurigenera of  $X$  vanish, and consequently  $X$  has to be ruled by the Enriques' criterion of ruledness.

Our main result is the following:

Theorem 2. Let  $X$  be a smooth projective ruled non-rational surface over  $\mathbb{C}$  with  $K^{-1}(X) = 2$ , and  $\pi: X \longrightarrow B$  the canonical ruling fibration of  $X$  (with  $B$  a smooth projective curve of genus  $g > 0$ ). Then there is a canonical birational morphism  $u: X \longrightarrow Y$ , with  $Y$  a normal projective surface having the following properties:

a)  $Y$  has precisely one non-rational singularity and (possibly) finitely many rational singularities; if  $X$  contains no redundant exceptional curves then  $X$  is the minimal desingularization of  $Y$ .

b)  $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$  and the geometric genus of the non-rational singularity is  $g$ .

c) The exceptional fibre of  $u$  over every rational singularity of  $Y$  is contained in a degenerated fibre of  $\pi$  (to different rational singularities it may correspond different degenerated fibres of  $\pi$ ), while the exceptional fibre of  $u$  over the non-rational singularity of  $Y$  consists of a section of  $\pi$  plus (possibly) some irreducible components of the degenerated fibres of  $\pi$ .

d) The anticanonical ring  $R^{-1}(X)$  is isomorphic to the anticanonical ring  $R^{-1}(Y) = \bigoplus_{m=0}^{\infty} H^0(Y, \omega_Y^{(-m)})$ , where  $\omega_Y^{(r)}$  is the  $r$ -th power of the dualizing sheaf of  $Y$  (i.e. the double dual of the  $r$ -th tensor power of  $\omega_Y$  if  $r \geq 0$ , and  $\text{Hom}_{\mathbb{C}}(\omega_Y^{(-r)}, \mathcal{O}_Y)$  if  $r < 0$ ).

e) The anticanonical ring  $R^{-1}(X)$  is a finitely generated  $\mathbb{C}$ -algebra if and only if there is a positive integer  $r > 0$  such that  $rK_Y$  is a Cartier divisor on  $Y$ , where  $K_Y$  is a canonical divisor on  $Y$  (in other words, iff  $\omega_Y^{(r)}$  is invertible for some  $r > 0$ ).

f) If  $rK_Y$  is a Cartier divisor for some  $r > 0$ , then  $-rK_Y$  is an ample divisor,

and in this case  $Y = \text{Proj}(R^{-1}(X))$ .

Conversely, if Y is a normal projective surface having at least a non-rational singularity and such that  $-rK_Y$  is an ample Cartier divisor for some  $r > 0$ , then the minimal desingularization X of Y is a ruled non-rational surface with  $K^{-1}(X) = 2$ . Moreover, the desingularization morphism  $u: X \rightarrow Y$  coincides (up to isomorphism) to the canonical morphism from the first part of the theorem, and hence Y enjoys all the above properties.

If X is a surface as in theorem 2, the surface Y will be referred to as the anticanonical model of X. As far as the proof of theorem 2 is concerned, it should be noted that it relies first of all on the Zariski decomposition of the anticanonical divisor of X (see [15]) together with some ideas developed by Sakai in [12], and secondly, on a nice extension due to Brenton (see [4], and also proposition 5 below) of a contractibility criterion of Artin (see [1]).

As an illustration of what theorem 2 means, take  $X = P(0_B \oplus L^{-1})$ , with B a smooth projective curve of genus  $g > 0$ , and L a line bundle on B of degree  $e > 2g - 2$ . As we have already remarked, Sakai showed that  $K^{-1}(X) = 2$ . Then theorem 2 implies that Y is the projective cone over the polarized curve (B, L) and u is the canonical morphism which blows down the minimal section of  $\tilde{\pi}$  (i.e. the unique section E with  $E^2 = -e$ ). Moreover, one can see that  $rK_Y$  is a Cartier divisor for some  $r > 0$  iff  $L^{2g-2} \otimes \omega_B^{-e}$  is a torsion. Hence  $R^{-1}(X)$  is a finitely generated  $\mathbb{C}$ -algebra iff the above line bundle is a torsion.

We would like to thank Professor F. Sakai for sending us his preprint [12].

### §1. Preliminaries

Let X be a smooth projective surface with  $K^{-1}(X) = 2$ . Then it makes sense to consider the Zariski decomposition of the anticanonical divisor of X

$$-K = P + N, \text{ such that}$$

- P is a  $\mathbb{Q}$ -divisor which is numerically effective,
- N is an effective  $\mathbb{Q}$ -divisor such that either  $N = 0$ , or if  $N > 0$ , the intersection matrix of  $\text{Supp}(N)$  is negative definite,

- if E is an irreducible component of  $\text{Supp}(N)$  then  $P.E = 0$  (see [15], [12]).

Such a decomposition is unique and P (resp. N) is called the positive (resp. negative) part of  $-K$ . We refer to [12] for the properties of this decomposition.

Lemma 3 (see [12], table 3.3).  $K^{-1}(X) = 2$  if and only if  $P^2 > 0$ .

Definition (see [12]). An exceptional curve of the first kind  $E$  on  $X$  is said to be redundant if  $P.E = 0$ .

Proposition 4 (see [12], proposition 4.2). Let  $X$  be a smooth projective surface with  $\kappa^{-1}(X) = 2$ . Then  $X$  birationally dominates a smooth projective surface  $X_0$  with  $\kappa^{-1}(X_0) = 2$  and without redundant exceptional curves. Furthermore we have an isomorphism of graded  $\mathbb{C}$ -algebras  $R^{-1}(X) \cong R^{-1}(X_0)$ .

Although in [12] this proposition is stated only for rational surfaces, the same proof works also for non-rational surfaces.

Proposition 5 (see Composite theorem 1 of [4], and lemma 9 of [5]). Let  $Y$  be a normal compact complex surface and  $u: X \longrightarrow Y$  its minimal desingularization. Assume that  $X$  is projective algebraic and that  $p_g(Y) = p_g(X)$ . Then  $Y$  is projective algebraic.

The geometric genus  $p_g(Y)$  of a compact normal surface is defined by  $p_g(Y) = h^2(O_Y)$ . If  $y \in Y$  is a singular point of  $Y$  then the geometric genus of the singularity  $(Y, y)$  is by definition  $\dim_{\mathbb{C}}(R^1 u_* (O_X))_y$ , where  $u: X \longrightarrow Y$  is any desingularization of  $(Y, y)$ .

Proposition 6 (Zariski). Let  $X$  be a smooth projective surface with  $\kappa^{-1}(X) = 2$ . Then the anticanonical ring  $R^{-1}(X)$  is a finitely generated  $\mathbb{C}$ -algebra if and only if there is a positive integer  $r > 0$  such that  $rK$  is an integral divisor and the linear system  $|rK|$  has no fixed components. Moreover, if the latter condition holds one can even find a positive integer  $r$  such that  $rK$  is an integral divisor and the linear system  $|rK|$  has no base points.

Proposition 6 is a special case of [15], theorem 8a) (page 612) and theorem 6.1.

## §2. Proof of Theorem 2

Let  $A$  be the union of all integral curves  $E$  on  $X$  such that  $P.E = 0$ . Then  $\text{Supp}(N) \subseteq A$  and lemma 3 and Hodge index theorem imply that  $A$  contains only finitely many curves and that the intersection matrix of  $A$  is negative definite. Then by a theorem of Grauert (see [8]) there exists an analytic contraction of the curve  $A$ ; more precisely, there exists a morphism  $u: X \longrightarrow Y$  such that:

- $Y$  is a normal compact complex surface,
- if  $A_1, \dots, A_n$  are the connected components of  $A$ , then  $u(A_i)$  is a point  $y_i \in Y$ ,  $i = 1, \dots, n$ , and  $u$  is an isomorphism between  $X - A$  and  $Y - \{y_1, \dots, y_n\}$ .

Let  $Z$  be an effective divisor with support contained in  $A$ . Then  $K+Z = -P-N+Z$ , and hence  $(K+Z).P = -P^2 < 0$  (lemma 3). Since  $P$  is numerically effective, this implies

$$(1) \quad H^0(X, O_X(m(K+Z))) = 0 \text{ for every } m \geq 1 \text{ and } Z > 0 \text{ with } \text{Supp}(Z) \subseteq A.$$

On the other hand, the exact sequence

$$0 \longrightarrow O_X(K) \longrightarrow O_X(K+Z) \longrightarrow \omega_Z \longrightarrow 0$$

yields the exact sequence

$$0 \xrightarrow{(1)} H^0(O_X(K+Z)) \longrightarrow H^0(\omega_Z) \longrightarrow H^1(O_X(K)).$$

Applying duality of  $Z$  and  $X$  we then get

$$(2) \quad h^1(O_Z) \leq h^1(O_X) = g.$$

By the Zariski theorem of holomorphic functions we have

$$R^1 u_{*}(O_X) = \text{inv} \lim_Z H^1(O_Z),$$

and since the maps  $H^1(O_Z) \longrightarrow H^1(O_{Z'})$  (with  $Z \geq Z'$ ) are all surjective, from

(2) we get

$$(3) \quad \dim_{\mathbb{C}} R^1 u_{*}(O_X) \leq g,$$

i.e. the sum of the geometric genera of the singular points  $y_1, \dots, y_n$  of  $Y$  is less than or equal to  $g$ . Therefore we distinguish two cases:

Case 1.  $Y$  has at most rational singularities.

Then by a result of M. Artin (see [1])  $Y$  is a normal projective surface and a multiple  $rK_Y$  of the canonical divisor of  $Y$  is a Cartier divisor ( $r > 0$ ). According to the proof of theorem 1 of Sakai ([2], theorem 4.3), let  $r$  be such that  $r\Delta_i$  is an integral divisor ( $i = 1, \dots, n$ ), where the  $\mathbb{Q}$ -divisor  $\Delta_i$  is definite by the conditions

$$\Delta_i \cdot E = -K \cdot E \quad \text{for every irreducible component } E \text{ of } A_i, \text{ and}$$

$$\text{Supp}(\Delta_i) \subseteq A_i.$$

Then  $N = \sum \Delta_i$  and we have

$$u^*(rK_Y) \sim rK + rN = -rP,$$

and therefore  $u^*(-rK_Y) = rP$ . Using the argument in Sakai's proof (loc. cit.) and the Nakai-Moishezon criterion of ampleness, we easily infer that  $-rK_Y$  is an ample Cartier divisor. Therefore  $Y$  is a normal projective surface with only rational singularities and  $-rK_Y$  is an ample Cartier divisor for some  $r > 0$ . Then by theorem 1

X has to be a rational surface, a contradiction. su

Therefore case 1 does not occur.

Case 2. Y has at least one non-rational singularity.

Then we use:

Lemma 7. If  $A_i$  is contained in a degenerated fibre of  $\pi$ , then the singularity  $(Y, y_i)$  is rational.

Accepting for the moment lemma 1, we infer that there exists an irreducible component E of A which is not contained in any fibre of  $\pi$ . Then E dominates (via the morphism  $\pi$ ) the curve B, and in particular,  $p_a(E) = h^1(O_E) \geq g$ . Taking into account of (3) we get that  $p_a(E) = g$  and E is smooth. If  $g \geq 2$  then the Hurwitz's formula immediately implies that E is a section of  $\pi$ . If instead  $g = 1$  we also deduce that E is a section of  $\pi$  using essentially (1) and the so-called "Addition Formula" of the logarithmic Kodaira dimension (see e.g. [13], lemma 12).

Therefore E is a section of  $\pi$ . We claim that E is the only irreducible component of A which is not contained in any fibre of  $\pi$ . Indeed, suppose that E' is another irreducible component of A not contained in any fibre of  $\pi$ . As above E' has to be a section of  $\pi$ , and in particular  $p_a(E') = g$ . If E and E' belong to different connected components of A, we get a contradiction using the inequality (3).

Assume therefore that E and E' belong to the same connected component  $A_i$  of A. If  $E \cdot E' > 0$  then  $p_a(E+E') = p_a(E) + p_a(E') + E \cdot E' - 1 \geq p_a(E) + p_a(E') = 2g$ , and hence  $h^1(O_{E+E'}) \geq 2g$ , which contradicts (2). If  $E \cdot E' = 0$  and  $g \geq 2$  we get in the same way  $h^1(O_{E+E'}) \geq 2g-1$ , which again contradicts (2). If  $E \cdot E' = 0$  and  $g = 1$ , there is a sequence  $E_0 = E, E_1, \dots, E_p = E'$  ( $p \geq 2$ ) of irreducible components of  $A_i$  such that  $E_i \cdot E_{i+1} > 0$  ( $i = 0, 1, \dots, p-1$ ), since  $A_i$  is connected. Then using induction on i we get that  $p_a(E_0 + \dots + E_{p-1}) \geq 1$ , and hence  $p_a(E_0 + \dots + E_p) = p_a(E_0 + \dots + E_{p-1}) + p_a(E_p) + (E_0 + \dots + E_{p-1}) \cdot E_p - 1 \geq p_a(E_0 + \dots + E_{p-1}) + p_a(E_p) \geq 1+1 = 2$ , which violates once more (2).

Therefore we have shown that precisely one irreducible component E of A is a section of  $\pi$ , while all the other components of A are contained in the (degenerated) fibres of  $\pi$ .

Consider now the exact sequence arising from the Leray spectral sequence of u

$$(4) \quad 0 \longrightarrow H^1(O_Y) \longrightarrow H^1(O_X) \xrightarrow{\mathcal{E}} R^1 u_* (O_X) \longrightarrow H^2(O_Y) \longrightarrow H^2(O_X) = 0.$$

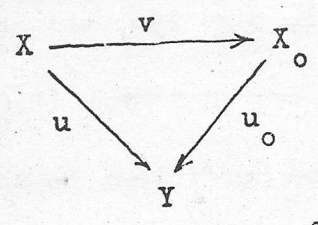
Identifying  $R^1 u_* (O_X)$  to  $\text{inv}_Z \lim H^1(O_Z)$ , then the map  $\mathcal{E}$  is induced by the restriction maps

$$\mathcal{E}_Z : H^1(O_X) \longrightarrow H^1(O_Z).$$

Taking  $Z = E$  we get that  $\mathcal{E}_E$  is an isomorphism. This is because  $E$  is a section of  $\pi$  and the canonical map  $\pi^*: H^1(O_B) \rightarrow H^1(O_X)$  is an isomorphism. Since  $\dim R^1 u_* (O_X) = g$  and the maps  $H^1(O_Z) \rightarrow H^1(O_{Z'})$  ( $Z \geq Z'$ ) are surjective, we infer that the map  $\mathcal{E}$  is an isomorphism, and therefore the exact sequence (4) yields  $h^1(O_Y) = h^2(O_Y) = 0$ . In particular, since  $h^2(O_X) = h^2(O_Y) = 0$ ,  $Y$  is projective algebraic by proposition 5.

Therefore, modulo lemma 7, we have proved parts a), b) and c) of theorem 2.

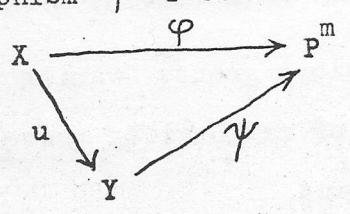
In order to prove part d), we apply proposition 4 to decompose the morphism  $u$  as in the following commutative diagram



with  $v$  a birational morphism,  $X_0$  smooth,  $R^{-1}(X) \cong R^{-1}(X_0)$  and  $X_0$  without redundant exceptional curves. Then  $u_0$  is nothing else than the minimal desingularization of  $Y$ . But as a consequence of a vanishing theorem due to Laufer and Ramanujan (see [11], [3]) we have  $(u_0)_* (\omega_{X_0}^{-m}) \cong \omega_Y^{(-m)}$  for every  $m \geq 0$  (see [12], lemma 1.6). This implies that  $R^{-1}(X_0)$  is isomorphic to  $R^{-1}(Y)$ , and therefore we get d).

Assume now that  $rK_Y$  is a Cartier divisor for some  $r > 0$ . As in the proof of case 1 we get that  $u^*(-rK_Y) = rP$ , which together the Nakai-Moishezon criterion of ampleness and a standard argument (cf. the proof of theorem 4.3 of [12]) show that  $-rK_Y$  is an ample Cartier divisor. Then a consequence of some results of Zariski (see [12], proposition 1.17) shows that  $R^{-1}(X) \cong R^{-1}(Y)$  is a finitely generated  $\mathbb{C}$ -algebra, and moreover,  $Y \cong \text{Proj}(R^{-1}(X))$ .

Conversely, assume that  $R^{-1}(X)$  is a finitely generated  $\mathbb{C}$ -algebra. Then by proposition 6 there is an integer  $r > 0$  such that  $rP$  is an integral divisor and the linear system  $|rP|$  has no base points. Let then  $\varphi: X \rightarrow P^m$  be the associated morphism into a projective space, with  $m = \dim |rP| - 1$  and  $\varphi^*(O_P(1)) \cong O_X(rP)$ . If  $C$  is an irreducible curve on  $X$  contracted by  $\varphi$  to a point then  $(rP) \cdot C = 0$ , or else  $P \cdot C = 0$ . In other words,  $C$  is a component of  $A$ . Since  $Y$  is normal (and hence  $u_*(O_X) = O_Y$ ), the morphism  $\varphi$  factors as



Denote by  $L = \psi^*(O_{P^m}(1))$ . Then  $u^*(L) \cong O_X(rP)$ . Moreover, we have:

$$u'^*(L/Y-\{y_1, \dots, y_n\}) \cong \mathcal{O}_X(rP)/X-A \cong \mathcal{O}_X(-rK)/X-A \cong u'^*(\omega_Y^{(-r)}),$$

with  $u'$  the isomorphism induced by  $u$  between  $X-A$  and  $Y-\{y_1, \dots, y_n\}$ . We infer that  $L/Y-\{y_1, \dots, y_n\} = \omega_Y^{(-r)}/Y-\{y_1, \dots, y_n\}$ . Since  $L$  and  $\omega_Y^{(-r)}$  are both reflexive sheaves of rank one, from this isomorphism we get that  $L$  and  $\omega_Y^{(-r)}$  are isomorphic, and thus  $\omega_Y^{(r)}$  is invertible.

In order to finish the proof of theorem 2 we have only to prove lemma 7.

Proof of lemma 7. Denote by  $D$  the (degenerated) fibre  $\pi^{-1}(b)$  containing  $A_i$ . We claim that  $H^1(\mathcal{O}_D) = 0$ . In order to prove the claim observe that by the base-change theorem (see [9], theorem 12.11, page 290) the canonical base-change map

$$R^1 \pi_* (\mathcal{O}_X) \otimes k(b) \longrightarrow H^1(D, \mathcal{O}_D)$$

is an isomorphism. Therefore it will be sufficient to show that  $R^1 \pi_* (\mathcal{O}_X) = 0$ . The latter equality is obvious if  $X = P(E) \longrightarrow B$  is a relatively minimal model, with  $E$  a rank two vector bundle over  $B$ . If  $X$  is not a relatively minimal model, let  $C$  be an exceptional curve of the first kind on  $X$ , and  $X'$  the smooth surface obtained by blowing down  $C$ . Let  $S = P(E)$  be a relatively minimal model dominated by  $X'$  and  $\pi': X' \longrightarrow B$  the canonical projection. By induction on the number of blowing ups entering in the composition of  $\pi$ , we may assume that  $R^1 \pi'_* (\mathcal{O}_{X'}) = 0$ . Since the morphism  $v: X \longrightarrow X'$  is a blowing up morphism, it is well known (and easy to see) that  $R^1 v_* (\mathcal{O}_X) = 0$ . From these equalities and from the Leray spectral sequence of the composition  $\pi = \pi' \circ v$  we get that  $R^1 \pi_* (\mathcal{O}_X) = 0$ , as required.

From  $H^1(\mathcal{O}_D) = 0$  and from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-nD) \otimes \mathcal{O}_D = \mathcal{O}_D \longrightarrow \mathcal{O}_{(n+1)D} \longrightarrow \mathcal{O}_{nD} \longrightarrow 0$$

we get by induction on  $n$  that  $H^1(\mathcal{O}_{nD}) = 0$  for every  $n \geq 1$ . Now, if  $Z$  is an effective divisor whose support is contained in  $A_i$ , there is a positive  $n \geq 1$  such that  $nD \geq Z$ , and hence we get the exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{nD} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

with  $L$  a sheaf supported by the curve  $nD$ . Taking cohomology we get the exact sequence

$$H^1(\mathcal{O}_{nD}) \longrightarrow H^1(\mathcal{O}_Z) \longrightarrow H^2(nD, L).$$

Since the first and the last cohomology groups vanish we get  $H^1(\mathcal{O}_Z) = 0$ , which proves lemma 7 using Artin's criterion of rationality (see [1], [2]).

Theorem 2 is thereby completely proved.

§3. Some comments and an application

First we want to indicate how one gets a better version of theorem 2 if we replace the complex field  $\mathbb{C}$  by the algebraic closure of a finite field.

Theorem 2'. The hypotheses are the same as in theorem 2 except the fact that  $\mathbb{C}$  is replaced by the algebraic closure  $k$  of a finite field. Then the same conclusions as in theorem 2 holds, and moreover the properties e) and f) can be replaced by the stronger one:

g) The anticanonical ring  $R^{-1}(X)$  is always a finitely generated  $k$ -algebra and  $Y = \text{Proj}(R^{-1}(X))$ . In particular, there always exists a positive integer  $r > 0$  such that  $-rK_Y$  is an ample Cartier divisor.

The proof of theorem 2' is practically the same as the proof of theorem 2, but using the following observations. First of all, over  $k$  we have a very nice contractibility criterion due to M. Artin (see [1], theorem 2.9), namely: if  $A$  is a curve on a smooth projective surface  $X$  such that the intersection matrix is negative definite, then there is a birational morphism  $u: X \longrightarrow Y$  with the following properties:

- $Y$  is a normal projective surface.
- If  $A_1, \dots, A_n$  are the connected components of  $A$  then  $u(A_i)$  is a point  $y_i \in Y$  for every  $i = 1, \dots, n$ .
- $u$  defines an isomorphism between  $X - A$  and  $Y - \{y_1, \dots, y_n\}$ .

Therefore, instead of using proposition 5 and Grauert's contractibility criterion, we simply use this criterion of M. Artin.

Secondly, if  $k$  is the algebraic closure of a finite field, then the divisor class group of every singularity  $(Y, y_i)$  is a torsion group. In particular, we can always find a positive integer  $r > 0$  such that  $rK_Y$  is a Cartier divisor on  $Y$ .

Remarks. 1. In the situation of theorem 2, the non-rational singularity of  $Y$  is however pararational in the sense of [5], as one can easily see.

2. One can ask the following question: which smooth projective surfaces  $X$  with  $K^{-1}(X) = 2$  have a Gorenstein anticanonical model? That is, classify all surfaces  $X$  with  $K^{-1}(X) = 2$  whose anticanonical model  $Y$  have the dualizing sheaf  $\omega_Y$  invertible (and by theorems 1 and 2, then necessarily  $\omega_Y^{-1}$  ample). This classification is already known (see [6], [7], [10]) and, roughly speaking, is the following:  $X$  is either a smooth rational surface which is obtained by blowing up  $d$  points of  $P^2$

( $1 \leq d \leq 8$  and infinitely near points allowed) in almost general position (in the sense of [7]), or  $X$  is a geometrically ruled surface  $X = P(E) \longrightarrow B$  over an elliptic curve  $B$ , with  $E = \mathcal{O}_B \oplus L^{-1}$  and  $\deg(L) > 0$ . In the first situation  $Y$  is either smooth and  $X = Y$  is a Del Pezzo surface, or  $Y$  is singular with only rational double points as singularities (a degenerated Del Pezzo surface). In the second situation  $Y$  is the projective cone over the elliptic polarized curve  $(B, L)$ .

More generally, theorems 1 and 2 allow us to say something about the structure of the normal projective surfaces  $Y$  for which there is a positive integer  $r > 0$  such that  $-rK_Y$  is an ample Cartier divisor.

Corollary 8. Let  $Y$  be a normal projective surface over  $\mathbb{C}$  such that  $-rK_Y$  is an ample Cartier divisor for some  $r > 0$ . Let  $u: X \longrightarrow Y$  be the minimal desingularization of  $Y$ . Then  $K^{-1}(X) = 2$ , the anticanonical ring  $R^{-1}(X)$  is a finitely generated  $\mathbb{C}$ -algebra,  $Y = \text{Proj}(R^{-1}(X))$ , and  $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$ . Moreover, one has one of the following possibilities:

- i)  $X$  is rational and  $Y$  has only rational singularities, or
- ii)  $X$  is ruled non-rational and  $Y$  has precisely one non-rational singularity and (possibly) finitely many rational singularities. These singularities are described in theorem 2, condition c).

This corollary is a direct consequence of theorems 1 and 2. Note that it implies the following: if  $Y$  is a normal projective surface over  $\mathbb{C}$  with only rational singularities (in particular, if  $Y$  is smooth) and such that  $-rK_Y$  is an ample Cartier divisor for some  $r > 0$ , then  $Y$  is a rational surface.

Examples of non-rational normal projective surfaces  $Y$  with  $-rK_Y$  an ample Cartier divisor for some  $r > 0$  are given by the following:

Proposition 9. Let  $B$  be a smooth projective curve of genus  $g > 0$  and  $L$  a line bundle of degree  $e > 2g - 2$ . If there exists a positive integer  $n > 0$  such that  $L^{n(2g-2)} \otimes \omega_B^{-ne} \cong \mathcal{O}_B$ , then  $-rK_Y$  is an ample Cartier divisor, with  $Y$  the projective cone over the polarized curve  $(B, L)$  and  $r = ne$ .

The proof of proposition 9 is standard and we omit it. Note that the following converse is also true:  $-rK_Y$  is not a Cartier divisor for every  $r > 0$  if the sheaf  $L^{2g-2} \otimes \omega_B^{-e}$  is not a torsion (see [14]).

Remark. Let  $X$  be a smooth projective ruled non-rational surface with  $K^{-1}(X) = 2$ . Let  $C$  be an integral curve with  $C^2 < 0$  and  $p_a(C) > 0$ . We claim that  $C$  coincides to

the section  $E$  of  $\pi$  which is a component of the exceptional fibre of  $u$  over the non-rational singularity of the anticanonical model of  $Y$  of  $X$ .

Indeed, according to the proof of theorem 2, it will be sufficient to show that  $P.C = 0$ , where  $-K = P + N$  is the Zariski decomposition. But the fact that  $p_a(C) > 0$  and the genus formula imply that  $K.C > 0$ , or else  $P.C + N.C < 0$ . Since  $P$  is numerically effective,  $P.C \geq 0$ , and hence  $N.C < 0$ . Since  $N$  is effective we infer that  $C$  must be a component of  $N$ , and so  $P.C = 0$ .

As a consequence we get that the only integral curves of  $X$  with negative self-intersection are: the section  $E$  of  $\pi$  and the irreducible components of the degenerated fibres of  $\pi$ .

Besides the examples of geometrically ruled surfaces with anticanonical dimension already mentioned above, one can easily provide further examples of ruled surfaces  $X$  with  $\kappa^{-1}(X) = 2$  in the following way.

First start with a surface  $X = P(0_B \oplus L^{-1})$ , with  $e = \deg(L) > 2g-2$ . Using the formula  $\omega_X = 0_X(-2) \otimes (\omega_B \otimes L^{-1})$  and the projection's formula one easily gets

$$p_{-n}(X) = \sum_{i=0}^{2n} h^0(B, L^{n-i} \otimes \omega_B^{-n}), \text{ for every } n \geq 1.$$

Since  $\deg(L^{n-i} \otimes \omega_B^{-n}) = (n-i)e - n(2g-2)$ , the Riemann-Roch inequality yields:

$$(5) \quad p_{-n}(X) \geq \frac{n(n+1)}{2} (e-4g+4) + (n+1)(1-g).$$

On the other hand, let  $X$  be an arbitrary ruled surface and  $\sigma: X' \rightarrow X$  the quadratic transformation of  $X$  of center  $x \in X$ , with the exceptional curve  $L = \sigma^{-1}(x)$ . The formula  $\omega_{X'} = \sigma^*(\omega_X) \otimes 0_{X'}(L)$  immediately yields the exact sequence ( $n \geq 1$ ):

$$0 \rightarrow \Gamma(\omega_{X'}^{-n}) \rightarrow \Gamma(\omega_X^{-n}) \rightarrow \Gamma(0_{nL}),$$

and since  $\Gamma(0_{nL}) \cong 0_{X,x} / m_x^n$  (where  $m_x$  is the maximal ideal of the local ring  $0_{X,x}$ ), we get:

$$(6) \quad p_{-n}(X') \geq p_{-n}(X) - \frac{n(n+1)}{2}.$$

Combining (5) with (6) we get:

Proposition 10. Let  $X = P(0_B \oplus L^{-1}) \rightarrow B$  be a geometrically ruled surface with  $e = \deg(L) > 4g-4$  and  $g \geq 1$ . Then the surface  $X'$  obtained from  $X$  by blowing up  $d$  points has  $\kappa^{-1}(X') = 2$  if  $d < e-4g+4$ .

Note. The inequality (5) is far from being sharp. On the other hand, if  $X$  is a ruled surface with  $\kappa^{-1}(X) = 2$  and  $R^{-1}(X)$  finitely generated over  $\mathbb{C}$ , one can get formulae for the antigenera of  $X$  along the same lines as in [12].

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