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ISSN 0250 3638

A TOPOLOGICAL CHARACTERIZATION OF CARTAN OPEN SUBSETS OF $\mathbb{R}\times\mathbb{C}$

by

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PREPRINT SERIES IN MATHEMATICS

No.23/1984

BUCUREȘTI

Med 20544

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April 1984

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1. Introduction

In [4] M. Jurchescu considers the notion of mixed manifold which represents a new point of view on the notion of differentiable family of complex manifolds considered by Kodaira-Spencer in their famous paper [5]. Among mixed manifolds, Cartan manifolds play an important rôle in the theory created by M. Jurchescu. In [2], the following characterization of Cartan open subsets of Cartan manifolds is obtained :

Let X be a Cartan manifold of type (m,n), let D be on open subset of X. D is itself a Cartan manifold (with respect to the induced mixed structure) iff $H^q(D, \mathcal{O}_D(\mathbb{C})) = 0$ for each $q = 1,2 \ldots n$.

Here $\mathcal{O}_X(\mathbb{C})$ denotes the canonical structure sheaf of a mixed manifold X (for the main properties of mixed manifolds see [4], [2]).

In the complex case, for (complex) dimension 1, there exists a topological characterization of the holomorphic convex hull $\hat{K}$ of a compact set $K\subset\mathbb{C}$, namely $\hat{K} = K\cup\{$set of relatively compact connected components of $\mathbb{C} - K\}$ (see eg.[ ]) and this result furnishes a tool which provides a proof to the fact that each open subset of $\mathbb{C}$ is Stein.

Example 1 in [2] shows that in the simplest similar mixed case $X = \mathbb{R}\times\mathbb{C}$, not every open subset of X is Cartan.

By the theorem quoted above, $D\subset\mathbb{R}\times\mathbb{C}$ is a Cartan open subset iff $H^1(D, \mathcal{O}_D(\mathbb{C})) = 0$. This condition is unfortunately not very easy to be verified even on simple examples.

The purpose of this paper is to give a topological characterization of Cartan open subsets of $\mathbb{R}\times\mathbb{C}$.

2. Some topological results in $\mathbb{R}^n$.

On $\mathbb{R}^n$ we shall always consider the canonical metric $d(z,u) = \left(\sum_{i=1}^{n}(z_i-u_i)^2\right)^{1/2}$ if $z=(z_1\dots z_n)$, $u=(u_1\dots u_n)$, and for $z\in\mathbb{R}^n$, we shall denote by $|z|$ the number $d(z,o)$.

<u>Definition 2.1.</u> Let F be a subset of $\mathbb{R}^n$ and let $K\subset F$. K is called isolated (in F) if there exists a relatively compact connected open subset Ω of $\mathbb{R}^n$ (which we shall denote by $\Omega\subset\subset\mathbb{R}^n$) such that $\Omega\supset K$, $\Omega\cap F\setminus K = \emptyset$. K is called weakly isolated (in F) if for each $x\in F\setminus K$, there exists $\Omega\subset\subset\mathbb{R}^n$ such that $\Omega\supset K$, $\Omega\not\ni x$, $\partial\Omega\cap F=\emptyset$.

Every isolated subset is weakly isolated, but the converse not true.

Example 2.1. If F is a convergent sequence in $\mathbb{R}^n$, together with its limit point K, K is weakly isolated, but not isolated. Let us remind that if $K\subset\mathbb{R}^n$, $x,y\in K$ and $\varepsilon>0$ we shall say (see [6]) that x and y are ε-connected if there exist $x_0=x, x_1\dots x_n=y$, $x_i\in K$ such that $d(x_i, x_{i+1})\le\varepsilon$. The set $\{x_i\}_i$ is called ε-chain.

The following result will be frequently applied throughout the paper.

Theorem 2.1. [6] A compact subset of $\mathbb{R}^n$ is connected iff each pair of its points are ε-connected for every $\varepsilon>0$.

The "$\Rightarrow$" implication is true with no compactness assumption.

The following result is a separation theorem which makes definition 2.1. meanigful.

Theorem 2.2 Let F be a closed subset of $\mathbb{R}^n$ and K a compact connected component of F. Let $K \subset U \subset\subset \mathbb{R}^n$. Then there exists an open subset Ω of U such that $\Omega \supset K$, $\partial\Omega \cap F = \emptyset$.

Proof.

Let $M := (\bar{U} \cap F) \cup \partial U$, M is a compact subset of $\mathbb{R}^n$ and $K \subset M$. We claim that K is a connected component of M; indeed, suppose $y \in K$ is in the same component of M with K. Fix $x \in K$. It follows x and y are ε-connected in M for any $\varepsilon > 0$. Since x and y are not in the same component of F, there exists $\delta > 0$ such that x and y are not ε-connected in F for $\varepsilon \leq \delta$. For any $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$, choose $x_1^n \ldots x_{i_n}^n \in M$ such that $d(x, x_1^n)$, $d(x_i^n, x_{i+1}^n)$ and $d(x_{i_n}^n, y)$ are smaller than $\frac{1}{n}$. By the definition of δ, among the x_i^n there are points of ∂U

Chose i_n such that $x_i^n \notin \partial U$ for $i \leq i_n$, $x_{i_n+1}^n \in \partial U$ and let $z_n = x_{i_n}^n$. Then $z_n \in F$ and $d(z_n, \partial U) \xrightarrow{n} 0$.

By passing (eventually) to a convergent subsequence, we may suppose $z_n \to z$. Then if follows that $z \in F \cap \partial U$ and moreover. z and x are ε-connected in F for any $\varepsilon > 0$. Therefore $z \in K$. This leads to a contractidion since $K \cap \partial U = \emptyset$.

Now by [6] (theorem of p.122.) there exists $\varphi : M \to E$ where E is the Cantor set on the line such that distinct connected components of M are separated by φ.

Extend φ to the whole space by Tietze's theorem. As K and ∂U are compact sets it follows that $\varphi(K)$ and $\varphi(\partial U)$ are compact subsets of the Cantor set and then there exists $\zeta, \eta \in \mathbb{R}$ such that $\varphi(K) \in (\zeta, \ )$, $\varphi(\partial U) \cap [\zeta, \eta] = \emptyset$.

Then we define $\Omega := \varphi^{-1}(\zeta, \eta) \cap U$ and it is easily checked that has the desired properties.

Corollary 2.2.1

Each compact connected component of a closed subset F of $\mathbb{R}^n$ is weakly isolated (in F).

Definition 2.2. Let X be a subset of $\mathbb{R}^3$. Considering the canonical projection $\pi : \mathbb{R}^3 \to \mathbb{R}$ $\pi(t,x,y)=t$, we shall denote by X_t the set $\{(x,y) \in \mathbb{R}^2 \mid (t,x,y) \in X\}$ and by $X_{(t)}$ the set $\{t\} \times X_t \subset \mathbb{R}^3$. When no confusion is possible, both these sets will be called "the fiber of X over t".

Definition 2.3. Let F be a connected closed subset of $\mathbb{R}^3$, $\pi(F) := I$. I is an interval and we suppose that it has more than two points. F is called a string, if, for each $t \in \mathring{I}$, F_t is a connected compact subset of $\mathbb{R}^2$ (Of course, this happens iff $F_{(t)}$ is a compact connected subset of $\mathbb{R}^3$). If the interval I has a closed end denoted by a, then F_a is called the a-end of the string.

Definition 2.4. If L is a subset of $\mathbb{R}^2$, if F is a string in $\mathbb{R}^3$ and if there exists $t \in \mathring{I}$ such that $F_t = L$ we say that F passes through L. If X is a closed subset of $\mathbb{R}^3$ and if I is an interval (with more than two points) and if for each $t \in I$ one can choose a compact connected component F_t of X_t such that $F := \bigcup_{t \in I} \{t\} \times F_t$ is a string, we shall say that the string is contained in X.

Some of the main properties of a string are given by

Proposition 2.1. Let $F \subset \mathbb{R}^3$ be a string $\pi(F) = I$, $\mathring{I} = (a,b)$

(i) if a is a closed end of I and F_a is compact then F_a is connected.

(ii) If a is a closed end of I, then F_a cannot have both compact and non compact connected components.

(iii) If $s<t$ and $[s,t]\subset(a,b)$ then $F_{|[s,t]} := F \cap \pi^{-1}([s,t])$ is a string

(iv) If $\varepsilon>0$, $s<t$ $[s,t]\subset(a,b)$, $\varepsilon<\min(s-a,\ b-t)$ then $\left(F_{|[s,t]}\right)_{\varepsilon} := \{ u\in\mathbb{R}^3 \mid d(u,F_{|[s,t]}) \leq \varepsilon\} \cap \pi^{-1}([s,t])$ is a string

(v) If $s<t$ and $[s,t]\subset(a,b)$, then $F_{|[s,t]}$ is a compact set in $\mathbb{R}^3$

<u>Proof</u>.

(i) Suppose F_a were not connected. Then there exist K_1 and K_2, compact connected components of F_a. By theorem 2.2, there exists $\Omega\subset\subset\mathbb{R}^2$, $\Omega\supset K_1$ $\partial\Omega\cap F_a=\emptyset$ $\Omega\cap K_2=\emptyset$.
As $\{a\}\times\partial\Omega$ is a compact subset of $D:=\mathbb{R}^3\setminus F$, there exists $\delta>0$ such that $[a,a+\delta]\times\partial\Omega\cap F=\emptyset$

By the fact that F is connected and F_t are also connected for each $t\in(a,a+\delta]$, it follows that either $F_t\subset\Omega$ or $F_t\subset \complement\Omega$ for all $t\in(a,\ a+\delta]$. In both cases, the existence of K_1 and K_2 furnishes a contractidicn to the fact that F is connected.

(ii) Follows immediately by a similar reasoning to that in (i) using theorem 2.2.

(iii) We have to show that $F_{|[s,t]}$ is connected and this follows immediately as one notices that the existence of open subsets $\Omega_1\subset\mathbb{R}^3, \Omega_2\subset\mathbb{R}^3$ with $\Omega_1\cup\Omega_2\supset F_{|[s,t]}$, $\Omega_i\cap F_{|[s,t]}\neq\emptyset$ $i=1,2.$ and $F_{|[s,t]}\cap\Omega_1\cap\Omega_2=\emptyset$, insure the existence of a point $r\in[s,t]$ such that $\pi(\Omega_1\cap F_{|[s,t]})\cap\pi(\Omega_2\cap F_{|[s,t]})\ni r$ and this contradicts the connectedness of $\left(F_{|[s,t]}\right)_r = F_r$

(iv) Let us denote $\left(F|_{[s,t]}\right)_\varepsilon$ by G.

G is a closed set, and for each $r\in[s,t]$

$$G_r = \bigcup_{|p-r|\leq\varepsilon} \{z\in\mathbb{R}^2 \mid d(z,F_p)\leq\sqrt{\varepsilon^2-(p-r)^2}\}$$

By (iii) $\bigcup_{|p-r|\leq\varepsilon} F_{(p)}$ is connected.

The projection $\pi_2 : \mathbb{R}^3 \to \mathbb{R}^2$ $\pi_2(t,x,y)=(x,y)$ is continous and hence $\bigcup_{|p-r|\leq\varepsilon} F_p$ is connected

On the other hand, as F_p is connected, for each $\delta>0$ the set $\{z\in\mathbb{R}^2 \mid d(F_p,z)\leq\delta\}$ is connected.

As for each p with $|p-r|\leq\varepsilon$, the connected set $\{z\in\mathbb{R}^3 \mid d(z,F_p)\leq\sqrt{\varepsilon^2-(p-r)^2}\}$ intersects the connected set $\bigcup_{|p-r|\leq\varepsilon} F_p$, it follows that G_r is connected.

We infer then immediately that G is a string.

(v) We show that $F_{[s,t]}$ is bounded. Suppose that there exists a sequence $(r_n, z_n)\in F|_{[s,t]}$ with $|z_n|\to\infty$. We may suppose $r_n\to r\in[s,t]$

If $M = \sup\{|z| \mid z\in F_r\}$, as F is a string, we have $M<\infty$

By the fact F is connected, for n sufficiently large, we get points (r_n, u_n) in F with $|u_n|\leq M$. As F_{r_n} is connected we obtain the existence of points (r_n, w_n) such that $w_n\in F_{r_n}\cap\{z\in\mathbb{R}^2 \mid M+1\leq|z|\leq M+2\}$.

It follows that $(w_n)_n$ converges to w_0 and as F is a closed set $(r_0, w_0)\in F$; hence $w_0\in F_{r_0}$ and $|w_0|\geq M+1$. This is a contractiction and the result follows.

Example 2.2. There exists a string whose a end has two noncompact connected components.

Let $F = \{ (t,x,y) \in \mathbb{R}^3 \mid 0 \leq t \leq 1,\ x = \pm 1 \Rightarrow y \in [0, \frac{1}{t}],$
$x \in (-1,1) \Rightarrow y = \frac{1}{t} \} \cup$

$\cup \{ (0,1,y) \in \mathbb{R}^3 \mid y \in \mathbb{R}_+ \} \cup \{ (0,-1,y) \in \mathbb{R}^3 \mid y \in \mathbb{R}_+ \}$

Definition 2.3 Let F be a closed subset of $\mathbb{R}^3$ let $t \in \pi(F)$ and let K_t be a compact connected of F_t.

K_t satisfies the property $(* +)$ (resp $(* -)$) if for each $\Omega \subset\subset \mathbb{R}^2$ such that $\partial\Omega \cap F_t = \emptyset, \Omega \supset K_t$, there exists $\delta > 0$ such that $[t, t+\delta] \times \partial\Omega \cap F = \emptyset$ (resp $[t-\delta, t] \times \partial\Omega \cap F = \emptyset$) and for each $s \in (t, t+\delta]$ (resp $[t-\delta, t)$), there exists a connected component K_s of F_s, with $K_s \subset \Omega$.

Let us note that by theorem 2.2. and by the fact that $\partial\Omega$ is compact and $\mathbb{R}^3 \setminus F$ is open, it follows that an Ω and a δ like in $(* +)$ resp $(* -)$ always exist and hence the definition is meaningful.

K_t is said to satisfy the property $(*)$ if it satisfies both $(* +)$ and $(* -)$.

F is said to satisfy the property $(* +)$ (resp$(* -)$, resp$(*)$) iff for each $t \in \overset{\circ}{\pi(F)}$, every compact connected component of the fiber of F over t satisfies $(* +)$ (resp$(* -)$, resp$(*)$).

Theorem 2.3. Let F be a closed subset of $\mathbb{R}^3$ which satisfies $(*)$. Then, for each $t \in \overset{\circ}{\pi(F)}$, and for each compact connected component K_t of F_t, there exists a string contained in F and which passes through K_t.

Proof. Let $s \in (a,b)$ we shall prove using only the property $(* +)$, that there exists a string E $\pi(E) = [s,t]$ $s<t$ such that the s-end of the string is K_s. The other half of

the proof is similar.

By theorem 2.2 there exists $\Omega \subset\subset R^2 \setminus \partial\Omega \cap F = \emptyset$ $\Omega \supset K_s$. As $\partial\Omega$ is compact there exists $t > s$ such that $[s,t] \times \partial\Omega \subset R^3 \setminus F$. The fibers of $G = F \cap [s,t] \times \Omega$ have then only compact connected components.

It is obvious that for each $r \in [s,t)$ each compact connected component K_r of (G_r) satisfies $(* +)$ (with respect to G).

Suppose now $x \in G_r$, $x' \in G_{r'}$ and $r \le r'$. For $\varepsilon > 0$ we shall say that x and x' are ε - order connected in G if there exist $x_i \in G_{r_i}$ $i = 1 \ldots n$ such that $d(x,x_1) \le \varepsilon$, $d(x_i, x_{i+1}) \le \varepsilon$ $d(x_n,x') \le \varepsilon$ and, moreover, $r \le r_1 \le \ldots \le r_n \le r'$.

If x and x' are ε-order connected in G for any $\varepsilon > 0$, we say that x and x' are order connected in G (in particular, they belong to the same connected component of G).

Fix now $x_0 \in G_s$; we shall prove first that there exists $x_1 \in G_t$ order connected in G to x_0.

For any $n \ge 1$, let $A_n = \{ r \in [s,t] \mid \exists y_r \in G_r \ \ y_r$ is $\frac{1}{n}$- order connected to $x_0\}$ and $r_n^* = \quad A_n$. By taking a convergent sequence of y_{r_k} $r_k \nearrow r_n^*$ it follows that $r_n^* \in A_n$. If $r_n^* < t$, take $y \in G_{r_n^*}$ $\frac{1}{n}$-connected to x_0. Let $L_{r_n^*}$ be the connected component of y in $G_{r_n^*}$. Take a $\frac{1}{2n}$ -neighbourhood of $L_{r_n^*}$ such that $\partial\Omega \cap G_{r_n^*} = \emptyset$ (see th. 2.2) and apply $(* +)$ to $L_{r_n^*}$ and Ω.

Then we find $z \in F_{r'}$, with $r_n^* < r' < r_n^* + \frac{1}{n}$ and $d(z, L_{r_n^*}) < \frac{1}{n}$.

Since (th.2.1) $L_{r_n^*}$ is $\frac{1}{n}$ - connected, it follows easily that z is $\frac{1}{n}$ - order connected to x_0 which contradicts the definition of r_n^*. So $r_n^* = t$

Let $y_n \in G_t$, y_n $\frac{1}{n}$-order connected to x_0. We may suppose that y_n has a limit point x_1; this x_1 is order connected to x_0.

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For $0 \leq \theta \leq 1$ define $r_\theta = (1-\theta)s+\theta t$. (then $r_0=s, r_1=t$). We show next that there exists $x_{\frac{1}{2}} \in G_{r_{1/2}}$ order connected to x_0 and to x_1. Indeed, suppose $y_1^n \dots y_{k_n}^n$ are chosen such that $y_i^n \in G_{r_i}$ $s \leq r_1 \quad \dots \leq r_{k_n} \leq t$, $d(x_0, y_1^n) < \frac{1}{n}$, $d(y_i^n, y_{i+1}^n) < \frac{1}{n}$, $d(y_{k_n}^n, x_1) < \frac{1}{n}$. Let $r_{i_n}^{(n)}$ be such that $r_{i_n}^{(n)} \leq r_{1/2} \leq r_{i_n+1}^{(n)}$.

The sequence $(y_{i_n}^n)$ has a convergent subse quence whose limit is $x_{1/2}$. It is easy to checkthat $x_{1/2} \in G_{r_{1/2}}$ and is order connected to x_0 and x_1,

It is now clear that we can apply induction in order to choose diadic rational $q(q=\frac{k}{2^m})$, a point $x_q \in G_{r_q}$, such that for, any $q \neq q'$ x_q is connected to $x_{q'}$, define E_{rq} to be the connected component of x_q in G_{r_q}.

Finally, if $\alpha \in (0,1)$ is any real number which is not a diadic rational, then take $q_n \to \alpha$, q_n diadic rational. Then x_{q_n} has a convergent subsequence to x_α.

Define E_{r_α} to be the connected component of x_α in G_{r_α}.

The definition is consistent : if x'_q is a sequence convergent to x'_α, E'_{r_α} is the connected component of x'_α in G_{r_α} and $E_{r_\alpha} \neq E'_{r_\alpha}$, we take by theorem 2.2 $\Omega \subset\subset R^2$

$E_{r_\alpha} \subset \Omega \quad E'_{r_\alpha} \cap \Omega = \emptyset \quad \partial\Omega \cap G_{r_\alpha} = \emptyset$

Then $x_{q_n} \in \Omega$, $x'_{q_n} \notin \Omega$, $\partial\Omega \cap G_{r_{q_n}} = \emptyset$, hence for q_n sufficiently close to α, contradicting the fact that x_{q_n} and x'_{q_n} are in the same component of $G_{r_{q_n}}$. A similar argument

shows that $E = \bigcup_{\alpha \in [0,1]} \{r_\alpha\} \times E_{r_\alpha}$ is closed and E is therefore a string.

Example 2.3

$$X = \{ (t,x,y) \in \mathbb{R}^3 \mid \ y=0 \ \ x \in [0,1], t=f_{(x)},$$

f continuons and nowhere differentiable $\}$

Then there exists no string contained in X

Indeed, suppose F is a string. Let $[c,d] = \pi(F)$. For any $s \in [c,d]$ F_s must be a point, say $\varphi(s)$. If F is closed, it follows easily that φ is continuous ; moreover, φ is injective.

It is therefore a monotone bijection from $[c,d]$ to $[a,b] \subset [0,1]$ and $\varphi^{-1}=f$. Then f is monotone on $[a,b]$ which is impossible by Lebesque's theorem the almost every where differentiability of monotone functions.

3. A topological characterization of Cartan open subsets of $\mathbb{R} \times \mathbb{C}$. In the following lines, we shall always consider the canonical topological identification of $\mathbb{R} \times \mathbb{C}$ with $\mathbb{R}^3$ $((t,z) \rightsquigarrow$ $\rightsquigarrow (t, \mathrm{Re}z, \mathrm{Im}z))$.

In [1] one can find the definition of the notion of regular family of complex manifolds. In our case the definition is the following :

Definition 3.1 Let D be an open subset $\mathbb{R} \times \mathbb{C}$. D is called regular if for each $t \in \pi(D)$, there exist $\delta > 0$ and an open subset V of $\mathbb{C}$ such that.

(i) The pair (V, D_t) is a Runge pair

(ii) $(t-\delta, t+\delta) \times V \supset \pi^{-1}((t-\delta, t+\delta)) \cap D$

Let us reunind that due to [3], a pair of open subsets of $\mathbb{C}$ (V, D_t) is a Runge pair iff $V \setminus D_t$ has no relatively compact (in V) connected components.

Lemma 3.1 Let D be an open subset of $R \times \mathbb{C}$ suppose $\pi(D) = (a,b)$ and suppose $(a,b) = \bigcup_{\alpha \in A} (a_\alpha, b_\alpha)$. If for each $\alpha \in A$, $D_\alpha := D \cap \pi^{-1}((a_\alpha, b_\alpha))$ is a Cartan open subset of $R \times \mathbb{C}$, then D is Cartan.

Proof. Suppose $(t_n z_n)$ is a discrete sequence in D. We have to find $f \in \Gamma(D, \vartheta_D(\mathbb{C}))$ such that $\sup_n |f(t_n, z_n)| = +\infty$

If is a discrete sequence in $\mathbb{R}$ we take $f(t,z) = t$. If $t_n \to a$ (resp b) and this value is finite the function $f(t,z) = \frac{1}{t-a}$ is the desired one.

If $t_n \to t_0 \in (a,b)$, there exists $\alpha \in A$, such that $t_0 \in (a_\alpha, b_\alpha)$ For n sufficiently large, $(t_n z_n) \in D_\alpha$. As D_α is a Cartan open subset of $\mathbb{R} \times \mathbb{C}$, there exists $q \in \Gamma(D_\alpha, \vartheta_{\mathbb{R}\times\mathbb{C}}(\mathbb{C}))$ for which $\sup_n |q(t_n, z_n)| = +\infty$.

Let us consider $\eta \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} \eta \subset (a_\alpha, b_\alpha)$, $\eta(s) = 1$ for s in a neighbourhood of t_0, $0 \leq \eta \leq 1$.

Then, by defining

$$F(t,z) = \begin{cases} \eta(t)\, q(t,z) & t \in \operatorname{supp} \eta \\ 0 & \text{otherwise} \end{cases}$$

we obtain a function, with provides a element $f \in \Gamma(D, \vartheta_{\mathbb{R}\times\mathbb{C}}(\mathbb{C}))$ when resticted to D.

It is ob vious that $\sup_n |f(t_n, z_n)| = +\infty$

<u>Proposition 3.1</u> Let F be a string in $\mathbb{R}\times\mathbb{C}$, $\overset{\circ}{\widehat{\pi(F)}} = (a,b)$ Then $D = (a,b) \times \mathbb{C} \setminus F$ is a Cartan open subset of $\mathbb{R}\times\mathbb{C}$

<u>Proof</u>

Let $t_o \in (a,b)$ let $[s,t] \subset (a,b)$, $t_o \in (s,t)$. Let $n_0 = \max\left(\left[\frac{1}{s-a}\right] + 1, \left[\frac{1}{b-t}\right] + 1\right)$ (the symbol $[x]$ denotes the biggest integerless or equal to x)

For each $n \geq n_0$ we consider the following construction of D_n

By proposition 2.1. (v) $F|_{[s,t]}$ is compact, hence there exists $0 < M < \infty$ such that $\sup_{\substack{z \in F_r \\ r \in [s,t]}} |z| \leq M.$

Set then

$$D_n := \{(r,z) \in \mathbb{R}\times\mathbb{C} \mid r \in (s,t),\ |z| < M+n\} \setminus \left(F|_{[s,t]}\right)_{\frac{1}{n}}$$

(see prop 2.1. (iv))

Then, (D_n) have the following properties

(1) D_n is an open subset of $\mathbb{R}\times\mathbb{C}$

(2) $\bar{D}_n \subset D_{n+1}$ for $n \geq n_0$ (the closure is considered in $(s,t)\times\mathbb{C}$)

(3) $\bigcup_{n \geq n_0} D_n = D \cap \pi^{-1}((s,t))$

(4) For $r \in (s,t)$, the fiber of $F_n := (s,t)\times\mathbb{C}\setminus D_n$ is given by

$$F_{n_r} = \bigcup_{|p-r| \leq \frac{1}{n}} \left\{ z \in \mathbb{C} \mid d(z,F_p) \leq \sqrt{\frac{1}{n^2} - (p-r)^2} \right\} \supseteq F_r$$

Moreover $F_{n_r} - F_r$ cannot be closed as F is connected

Considering then $V = \mathbb{C} \setminus F_r$, one sees that

(i) the pair (V, D_{n_r}) is a pair

(ii) $(r-\frac{1}{n}, r+\frac{1}{n}) \times V \supset D_n \cap \pi^{-1}((r-\frac{1}{n}, r+\frac{1}{n}))$

It follows then that D_n are regular for each $n \geq n_0$

By [1] and [2], D_n are Cartan open subsets of $\mathbb{R}\times\mathbb{C}$

(5) For any compact $K \subset (s,t)$, $K\times\mathbb{C} \cap \bar{D}_n$ is a compact set.

(6) For $n \geq n_0$, $r \in (s,t)$, the pair (D_{n+1_r}, D_{n_r}) is a pair and hence by [1] (prop.10 and lemma p.212) it follows that we may apply prop.9 pp.209 from [1], ~~and we~~ obtain

$$H^1(D \cap \pi^{-1}((s,t)), \mathcal{O}_{\mathbb{R}\times\mathbb{C}}(\mathbb{C})) = 0$$

By [2], $D \cap \pi^{-1}((s,t))$ is then a Cartan open subset of

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$\mathbb{R}\times\mathbb{C}$. By lemma 3.1, it follows that $\dot{D}$ is a Cartan open subset of $\mathbb{R}\times\mathbb{C}$.

The main result of the paper is :

<u>Theorem 3.1</u> Let F be a closed subset of $\mathbb{R}\times\mathbb{C}$, let

$D := \mathbb{R}\times\mathbb{C} \setminus F$

The following statements are equivalent

(i) D is Cartan

(ii) For each compact subset L of D, $\hat{L}_D$ is compact

(iii) F satisfies (*)

(iv) Through any compact connected component of any fiber F a string contained in F passes.

<u>Proof</u>

(i)$\Rightarrow$(ii)

(ii)$\Rightarrow$(iii) Take t $\overset{\circ}{\overline{\pi(F)}}$, K_t a compact connected component of F_t. Consider $\Omega \subset\subset \mathbb{C}$ such that $\partial\Omega \cap F_t = \emptyset$, $\Omega \supset K$. If there existed a sequence $t_n \to t$ such that $\Omega \cap F_{t_n} = \emptyset$, it follows that for n sufficiently large $\bar{\Omega} \cap F_{t_n} = \emptyset$, and

$L := \bigcup_n \{t_n\} \times \partial\Omega \ \cup \{t\} \times \partial\Omega$, one obtains that

$\hat{L}_D = \bigcup_n \{t_n\} \times \bar{\Omega} \quad \cup(\{t\} \times (\bar{\Omega} \setminus K))$ and hence not a compact set. and this contradicts (ii)

(iii)$\Rightarrow$(iv) by Theorem 2.2.

(iv)$\Rightarrow$(i) Let (t_n, z_n) be a discrete sequence in D. We have to find.

$f \in \Gamma(D, \mathcal{O}_{\mathbb{R}\times\mathbb{C}}(\mathbb{C}))$ such that $\sup_n |f(t_n, z_n)| = +\infty$

If (t_n) or (z_n) are discrete sequences one of the coordinate functions provide the desired f.

If $t_n \to$ a and a $\notin \pi(D)$ the function $f(t,z) = \frac{1}{t-a}$ is the one we were looking for.

Suppose now $t_n \to a$ and $a \in \pi(D)$ and $z_n \to w$. It follows $(a, w) \in F$.

First, if w belongs to a noncompact connected component L_a of F_a. Then $D = \mathbb{R}\times\mathbb{C} \setminus \{a\} \times L_a$ is regular and hence Cartan.

There exists $\tilde{f} \in \Gamma(D, \mathcal{O}_{\mathbb{R}\times\mathbb{C}}(\mathbb{C}))$ which is unbounded on (t_n, z_n), and by resticting $\tilde{f}$ to D, we obtain the desired f.

If, on the other hand w belongs to a compact connected component K_a of F_a, there exists a string S which passes through K_a and which is contained in F. By proposition 3.1 there exists $\delta > 0$ such that $(a-\delta, a+\delta)\times\mathbb{C} \setminus S$ is a Cartan open subset of $\mathbb{R}\times\mathbb{C}$. An argument similar to the one used in the proof of lemma 3.1 concludes then the proof.

Final remarks

Let us note that the notion of regular family of open subsets of $\mathbb{C}$ (def.3.1) is not invariant under isomorphisms $\varphi : \mathbb{R}\times\mathbb{C} \to \mathbb{R}\times\mathbb{C}$ while the notions of Cartan open subset of $\mathbb{R}\times\mathbb{C}$ and complementary of a string in $\mathbb{R}\times\mathbb{C}$ are invariant.

The topological tools developped here are useful also in the investigation of the continuation problem for mixed functions.

The study of this phenomenon will be the subject of a forthcoming paper.

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