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REMARKS ON TWO THEOREMS OF CILIBERTO AND SERNESI

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In this paper we work over an algebraically closed field k of arbitrary characteristic. We use the standard notations from [Ha 1]. By a *curve* (resp. *surface*) we mean a k-algebraic smooth and irreducible scheme of dimension 1 (resp. 2).

1 Introduction

In [P 2] we considered the Halphen-Castelnuovo problem, denoted for any integer $n \geq 2$ by

HC(n): For which pairs of integers (d, g), $d \geq n$, $0 \leq g \leq \pi_0(d, n)$ there is a (smooth, irreducible) curve $C \subset \boldsymbol{P}^n$, non-degenerate in $\boldsymbol{P}^n$, of degree d and genus g ?

We recall that a curve $C \subset \mathbf{P}^n$ is called *non-degenerate* if it is a not contained in any hyperplane. Here $\pi_0(d, n)$ is the first one from the Harris-Eisenbud numbers $\pi_p(d, n)$ (see [H], ch. 3: Castelnuovo theory), given (for $0 \leq p \leq n-2$) by:

$$\pi_p(d, n) = \frac{m_p(m_p - 1)}{2}(n + p - 1) + m_p(\varepsilon_p + p) + \mu_p, \tag{1.1}$$

where

$$m_p = m_p(d, n) = \left[\frac{d-1}{n+p-1}\right]_* \tag{1.2}$$

$$\varepsilon_p = \varepsilon_p(d, n) = d - 1 - m_p(n + p - 1) \tag{1.3}$$

$$\mu_p = \mu_p(d, n) = \max\left(0, \left[\frac{p - n + 2 + \varepsilon_p}{2}\right]_*\right) \tag{1.4}$$

(we denote, during this paper by $[x]_*$ the integer part of $x \in \mathbf{R}$).

We remark that $\mu_0 = 0$ and $\pi_p(d,n) = \dfrac{d^2}{2(n+p-1)} + O(d)$.

The first contribution to $HC(n)$ belongs to Castelnuovo who found in 1893 the so-called Castelnuovo bound, namely

Theorem 1.1 (Castelnuovo [C]): *If $n \in Z$, $n \geq 3$ and $C \subset P^n$ is a non-degenerate (smooth, irreducible) curve of degree d and genus g, then $d \geq n$ and $0 \leq g \leq \pi_0(d,n)$.*

This result is a generalization of a previous similar result obtained by Halphen and Noether in 1882 for $n = 3$ ([Hl], [N]). For a modern proof see [H], ch.3.

$HC(2)$ is simple and well-known: there is a (smooth, irreducible) curve $C \subset \mathbf{P}^2$ of degree d and genus g if and only if $d \geq 1$, $g = \dfrac{(d-1)(d-2)}{2}$ (for $d = 1$, C is degenerate).

But for $n \geq 3$, $HC(n)$ becomes highly non-trivial. A (correct) solution for $HC(3)$ has been proposed by Halphen in 1882 ([Hl]), but his proof was, partially, incorrect. A complete proof was given by Gruson and Peskine 100 years later ([GP1], [GP2]). Before giving the answer to $HC(3)$ a

Definition 1.2: *A pair $(d,g) \in Z \times Z$, $d \geq n$, $0 \leq g \leq \pi_0(d,n)$ is called a gap for HC(n) if there is no non-degenerate (smooth, irreducible) curve $C \subset P^n$ of degree d and genus g.*

As for $HC(3)$, there are two domains in the (d,g)-plane: a domain D_1^3 where there is no gap for $HC(3)$ (the non-lacunar domain) and another domain D_2^3 where there are gaps (the lacunar domain): a pair $(d,g) \in D_2^3$ *is not* a gap for $HC(3)$ if and only if $(\exists) a, b \in \mathbf{Z}$, $a, b \geq 0$ such that $d = a + b$, $g = (a-1)(b-1)$.

$HC(n)$ has been solved for $n = 4, 5$ by Rathmann ([Ra]), $n = 6$ by Ciliberto ([Ci]) and "almost" solved by Ciliberto if $n = 7$ ([Ci]). The situation is similar to the case $n = 3$ in the sense that there are two domains: D_1^n where there is no gap and D_2^n where there are gaps, $n \in \{3,4,5,6,7\}$.

In [P 2] we defined the non-lacunar domain D_1^n and the lacunar domain D_2^n for any $n \geq 3$ and we expained why we gave these definitions (see [P 2], §4). We recall here the definitions. Firstly, we recall from [P 2] the definition of some numerical functions $\alpha_p(d,n)$: for any $p, n \in \mathbf{Z}$, $p \geq 0$, $n \geq 3$:

$$\alpha_p(d,n) := \frac{x_p(x_p-1)}{2}(n+p-1) + x_p(t_p+p) + u_p, \tag{1.5}$$

where

$$x_p = x_p(d,n) := \left[\frac{d - a_p^n}{n+p-1}\right]_*, \quad a_p^n = \left[\frac{n-p}{2}\right]_* + 1 \tag{1.6}$$

$$t_p = t_p(d,n) := d - 1 - x_p(n+p-1) \tag{1.7}$$

$$u_p = u_p(d,n) := \left[\frac{p - n + 1 + t_p(d,n)}{2}\right]_*. \tag{1.8}$$

Moreover, we recall from [P 2] the functions:

$$d_1(n) := \begin{cases} \max\left(2n+1, \frac{1}{6}(3 + (4k-1)\sqrt{24k-33})\right), & n = 3k \\ \max\left(2n+1, \frac{4k}{4k+1}(4k + \sqrt{32k^3 + 16k^2 - 2k - 1})\right), & n = 3k+1 \\ \max\left(2n+1, 5k+3+(2k+1)\sqrt{48k+6}\right), & n = 3k+2 \end{cases} \tag{1.9}$$

where $k \in \mathbf{Z}$, $k \geq 2$, $k = \left[\frac{n}{3}\right]_*$,

$$B(d,n) := \begin{cases} \left.\begin{array}{l} \alpha_{k+1}(d,n), \text{ if } n = 3k+1, 3k+2 \\ \alpha_k(d,n), \text{ if } n = 3k \end{array}\right\} & \text{if } 2n+1 \leq d < d_1(n) \\ A(d,n), & \text{if } d \geq d_1(n) \end{cases} \tag{1.10}$$

where

$$A(d,n) := \begin{cases} \pi_k(d,n), & \text{if } n \equiv 0,1 \pmod 3 \\ \pi_k(d,n) - \mu_k + \max(0, \varepsilon_k - 3k - 1), & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

with k from before.

Then

$$D_1^n : 0 \leq g \leq \begin{cases} \pi_0(d,n), & n \leq d \leq 2n \\ B(d,n), & d \geq 2n+1 \end{cases} \tag{1.11}$$

$$D_2^n : B(d,n) < g \leq \pi_0(d,n),\ d \geq 2n+1. \tag{1.12}$$

D_1^n and D_2^n are domains in the (d,g)-plane representing the non-lacunar domain and the lacunar domain, respectively,for $HC(n)$.

In [P 2] we proved the following

Theorem T: *In the domain D_1^n there is no gap for the problem $HC(n)$, for any $n \geq 3$, $n \in Z$.*

Our main contribution to Theorem T was for $n \geq 8$ (the Theorem T has been proved for $3 \leq n \leq 7$ by Gruson-Peskine ($n = 3$) [GP 2], Rathmann ($n = 4, 5$) [Ra], Ciliberto ($n = 6, 7$) [Ci]) and for "small" degrees, $2n + 1 \leq d < D(n)$ (= a quadratic function in n) (the Theorem T has been proved for $d \geq D(n)$ by Ciliberto and Sernesi (see [Ci])).

During the proof of Theorem T, in [P 2] the domain D_1^n has been divided in subdomains A_p^n, namely:

$$A'^n_{n-3} : 0 \leq g \leq \alpha_{n-3}(d,n),\ d \geq 2n+1 \tag{1.13}$$

$$A_p^n : \alpha_{p+1}(d-1,n) \leq g \leq \alpha_p(d,n),\ d \geq 2n+1,\ \frac{n}{3} \leq p \leq n-4 \tag{1.14}$$

$$\tilde{A}_k^n : \alpha_{k+1}(d-1,n) < g \leq B(d,n),\ d \geq 2n+1,\ k = \left[\frac{n}{3}\right]_* \tag{1.15}$$

$$A_0^n : 0 \leq g \leq \pi_0(d,n),\ n \leq d \leq 2n. \tag{1.16}$$

Then, we proved in [P 2] that

$$D_1^n = A_0^n \bigcup A'^n_{n-3} \bigcup \left(\bigcup_{\frac{n}{3} \leq p \leq n-4} A_p^n \right) \bigcup \tilde{A}_k^n. \tag{1.17}$$

The absence of gaps for $HC(n)$ in D_1^n is now a consequence of the following three theorems:

Theorem A: *If $n \geq 8$, $n \in Z$ and $(d,g) \in \tilde{A}_k^n$, then there is a (smooth, irreducible) non-degenerate curve P^n of degree d and genus g.*

Theorem B: *If $n \geq 8$, $n \in Z$ and $(d,g) \in A'^n_{n-3}$, then there is a (smooth, irreducible) non-degenerate curve P^n of degree d and genus g.*

Theorem C: *If $n \geq 8$, $n \in Z$ and $(d,g) \in \bigcup_{\frac{n}{3} \leq p \leq n-4} A_p^n$, then there is a (smooth, irreducible) non-degenerate curve P^n of degree d and genus g.*

(the existence of curves in A_0^n being elementary).

In [P 2] we gave a complete proof of Theorem C and only some ideas of proofs for Theorems A and B. The aim of this paper is to give complete proofs of Theorem A (in §2) and Theorem B (in §3).

The Theorem A is, essentially, covered by Theorem 1.1 from [Ci]. But, because our statement, necessary in [P 2], differs from Ciliberto statement, it is necessary to prove it.

Similarly, the main part of the domain A'^n_{n-3} is covered by our Theorem 1 from [P 1] and by Ciliberto-Sernesi Main Theorem from [CS]. Again, because the statement of Theorem B, necessary in [P 2], differs from the previous two statements, it is necessary to give the proof of Theorem B.

For completeness we include here some results from [P 2] and from [Ci].

2 The proof of Theorem A

The proof consists in a comparison between the *upper bound* of genus from our Theorem C and the *lower* bound of genus from Ciliberto's Theorem ([Ci]). This comparison is possible due to the following Lemma, proved in [P 2].

Lemma 2.1 (Lemma 3.3.2 from [P 2]):

i) $\alpha_p(d,n) = \left[F_d^{p,n}(2(x_p(d,n)+1)) - \frac{1}{2}\right]_*$;

ii) $\alpha_{p+1}(d-1,n) = \left[F_d^{p,n+1}(2(x_{p+1}(d-1,n)+1))\right]_*$, where

$F_d^{p,n}(r) := \frac{1}{2}\left[d(r-1) + (p-1)r - \frac{n+p-1}{4}r^2\right] + 1.$

Let's recall from [P 2] the definition of some surfaces X_p^n; X_p^n are rational surfaces with hyperelliptic hyperplane sections. Let there be $\Sigma = \Sigma_p^n = \{P_0, P_1, \ldots P_{s_p^n}\} \subset \mathbf{P}^2$ a set of general points, $s_p^n := 3p - n + 5$. Then, put $S_p^n := Bl_{\Sigma_p^n}(\mathbf{P}^2)$. We have: $Pic(S_p^n) \cong \mathbf{Z} \oplus \mathbf{Z}^{s_p^n+1}$ having $(l; -e_1, -e_2, \ldots, -e_{s_p^n})$ as a $\mathbf{Z}$-basis, where l is the class of preimage of a line from $\mathbf{P}^2$ and e_i, $0 \le i \le s_p^n$ are the classes of the exceptional divisors corresponding to the points P_i. The intersection form on $Pic(S_p^n)$ is given by $l^2 = 1$, $(e_i \cdot e_j) = 0$, $(\forall) i \neq j$, $(l \cdot e_i) = 0$, $(\forall i)$, $(e_i^2) = -1$, $(\forall) i$. If $\mathcal{D} \in Pic(S_p^n)$, $\mathcal{D} = a\,l - \sum_{i=0}^{s_p^n} b_i e_i$, we write $\mathcal{D} = (a; b_0, b_1, \ldots, b_{s_p^n})$. If $\mathcal{D} = \mathcal{O}_{S_p^n}(D)$, $D \in Div(S_p^n)$, we denote by $[\mathcal{D}] = |D|$ the complete linear system associated to $\mathcal{D}$ (resp. D), so

$[\mathcal{D}] = [a; b_0, b_1, \ldots, b_{s_p^n}] = |D|$. By a result of Gimigliano ([Gi]; see also [P 3]), the linear system $[p+1; p, 1^{s_p^n}]$ is very ample on S_p^n for any $p \geq 1$. If φ_p^n is the closed embedding associated to $[p+2; p, 1^{s_p^n}]$ then take

$$X_p^n := \varphi_p^n(S_p^n), \quad X_p^n \cong S_p^n \tag{2.1}$$

(we write $(a; b^s)$ instead of $(a; \underbrace{b, \ldots, b}_{s \text{ times}})$).

It can be seen that $X_p^n \subset \mathbf{P}^n$ (a non-degenerate surface) and $\deg X_p^n = n + p - 1$. If $p = k = \left[\frac{n}{3}\right]_*$ we arrive to the surfaces X_p^n used in [Ci].

We recall now the Gruson-Peskine coordinates ([GP 2], step 2): if $\mathcal{D} = \mathcal{O}_{X_p^n}(D) \in Pic(X_p^n) \cong \mathbf{Z} \oplus \mathbf{Z}^{s_p^n+1}$, $D \in Div(X_p^n)$, $\mathcal{D} = (a; b_0, b_1, \ldots, b_{s_p^n})$, $a \geq b_0 \geq b_1 \geq \ldots \geq b_{s_p^n} \geq 0$, we consider the change of coordinates (in $Pic(X_p^n) \otimes \mathbf{Q}$):

$$r := a - b_0, \quad \theta_i = \frac{1}{2}r - b_i, \quad i = \overline{1, s_p^n}. \tag{2.2}$$

If $d := (D \cdot H_p^n)(H_p^n \in [\mathcal{H}_p^n],\ \mathcal{H}_p^n := (p+2; p, 1^{s_p^n}) \in Pic(X_p^n))$ and $g := p_a(\mathcal{D})$, we have:

$$d = \frac{n+p-5}{2}r + 2a + \sum_{i=1}^{s_p^n} \theta_i; \quad g = F_d^{p,n}(r) - \frac{1}{2}\sum_{i=1}^{s_p^n} \theta_i^2 \tag{2.3}$$

with $F_d^{p,n}(r)$ from Lemma 2.1.

The coordinates $(d, r; \theta_1, \ldots, \theta_{s_p^n})$ of $\mathcal{D}$ are called the Gruson-Peskine coordinates. Now, we recall from [P 2] the following

Proposition 2.2 (Proposition 3.2.3 from [P 2]; see also [P 3]): *Let there be $\mathcal{D} = \mathcal{O}_{X_p^n}(D) \in Pic(X_p^n)$, $\mathcal{D} = (a; b_0, b_1, \ldots, b_{s_p^n})$. If the conditions (C1), (C2), (C3), (C4), (C5) are simultaneously satisfied by $\mathcal{D}$ in the coordinates $(d, r; \theta_1, \ldots, \theta_{s_p^n})$, then $[\mathcal{D}] = |D| \neq \emptyset$, has no base point and contains a (smooth, irreducible) curve, where:*

$$\theta_i \equiv \frac{1}{2}r \pmod 1, \quad i = \overline{1, s_p^n}; \tag{C1}$$

$$d + \frac{1}{2}(p - n + 5) - \sum_{i=1}^{s_p^n} \theta_i \equiv 0 \pmod 2; \tag{C2}$$

$$(C3) \qquad |\theta_1| \leq \theta_2 \leq \ldots \leq \theta_{s_p^n} \leq \frac{r}{2};$$

$$(C4) \qquad -\theta_1 + \sum_{i=2}^{s_p^n} \theta_i \leq d - \frac{1}{2}(n-p+1)r;$$

$$(C5) \qquad d \geq (p-1)r + 2.$$

Remark 2.3: *If $s_p^n \leq 6$ (i.e. $p \leq \frac{n+1}{3}$) then (C5) follows from the previous (C1) - (C4).*

In order to prove Theorem A, it is necessary to recall some results from [Ci], necessary here. Firstly, let's remark that the functions $F_d^{p,n}(r)$ (from Lemma 1.2) achieves its maximum at the point

$$(2.4) \qquad \rho_p = \rho_p(d,n) = \frac{2(d+p-1)}{n+p-1}.$$

Lemma 2.4 (Lemma 1.11 from [Ci], see also [Ci] section 1.d):

a) *If $n = 3k+h$, $h \in \{0,1\}$ then there exist functions $s(d,n)$ and $\delta_0(n)$ such that $(\forall)d, g, r \in N$ with*

$$F_d^{k,n}(r-1) < g \leq F_d^{k,n}(r), \quad r \geq s(d,n),\ d \geq \delta_0(n)$$

there is a (smooth, irreducible) curve $C \subset X_k^n$ of degree d and genus g.

b) *If $n = 3k+2$ there exist functions $s(d,n)$, $\delta_0(n)$ such that $(\forall)d, g, r \in N$ with r odd and*

$$F_d^{k,n}(r-2) < g \leq F_d^{k,n}(r), \quad r \geq s(d,n),\ d \geq \delta_0(n)$$

there is a (smooth, irreducible) curve $C \subset X_k^n$ of degree d and genus g.

c) *The definitions of the functions $s(d,n)$ and $\delta_0(n)$ are:*

$$\delta_0(n) = \begin{cases} 2n+1, & n \neq 8 \\ 22, & n = 8. \end{cases}$$

$$s(d,n) = \begin{cases} \frac{1}{3}(-4k+1+\sqrt{12d+16k(k+2)-38}), & n = 3k, k \geq 3 \\ -2(2k+1)+2\sqrt{d+2(k-1)+(2k+1)^2}, & n = 3k+1, k \geq 3 \\ -(8k+3)+\sqrt{(8k+3)^2+24k+8d-11}, & n = 3k+2, k \geq 2 \end{cases}$$

We denote, following [Ci], section 1.g,

$$(2.5)\quad r_0 = r_0(d,n) := \begin{cases} \min\{r \in \mathbf{Z} | r \geq s(d,n)\}, \text{ if } n \equiv 0,1 \pmod 3 \\ \min\{r \in \mathbf{Z} | r \text{ odd}, \ r \geq s(d,n)\}, \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

Lemma 2.5 ([Ci], Lemma 1.16): *Let there be*

$$\delta_1(n) = \begin{cases} -k+1+\frac{1}{6}(4k-1)\sqrt{12k-27}, & n = 3k, k \geq 3 \\ -5k+1+4k\sqrt{k}, & n = 3k+1, k \geq 3 \\ \frac{1}{2}(-6k+1+\sqrt{(6k+1)^2+64k^3+11k^2+k}), & n = 3k+2, k \geq 2 \end{cases}$$

Then:

a) $d \geq \delta_1(n) \Rightarrow s(d,n) \leq \rho_k(d,n)$ (see (2.4))

b) $$\left.\begin{array}{l} F_d^{k,n}(r_0-1) \leq F_d^{k,n}(s(d,n)), \ n = 3k, 3k+1, k \geq 3 \\ F_d^{k,n}(r_0-2) \leq F_d^{k,n}(s(d,n)), \ n = 3k+2, k \geq 2 \end{array}\right\} (\forall) d \geq \delta_1(n)$$

Lemma 2.6 ([Ci], Lemma 1.18):

a) *If* $n = 3k, 3k+1$, *then* $[\max\{F_d^{k,n}(r) | r \in Z\}]_* = \pi_k(d,n)$.

b) *If* $n = 3k+2$, *then* $[\max\{F_d^{k,n}(r) | r \in Z, r \text{ odd}\}]_* = \pi_k(d,n) - \mu_k(d,n) + \max(0, \varepsilon_k(d,n) - 3k - 1)$ (see (1.1)-(1.4))

Let there be

$$(2.6)\qquad \bar{\varphi}(d,n) := \begin{cases} F_d^{k,n}(r_0(d,n)-1), \ n = 3k, 3k+1 \\ F_d^{k,n}(r_0(d,n)-2), \ n = 3k+2 \end{cases}$$

$$(2.7)\qquad d_0(n) := \max(\delta_0(n), \delta_1(n))$$

(see Lemmas 2.4, 2.5).

From Lemmas 2.5 and 2.6 Ciliberto deduces

Theorem 2.7 ([Ci], Theorem 1.1): *If* $d_0(n)$ *is as in (2.7), then* $(\forall) n, d, g \in Z$, $n \geq 6$, $d \geq d_0(n)$ *and*

$$\bar{\varphi}(d,n) < g \leq A(d,n) = \begin{cases} \pi_k(d,n), & n = 3k, 3k+1 \\ \pi_k(d,n) - \mu_k + \max(0, \varepsilon_k - 3k - 1), & n = 3k+2 \end{cases}$$

(see (1.1) - (1.4)) then there is a (smooth, irreducible) curve $C \subset X_k^n$, non-degenerate in $\mathbf{P}^n$, of degree d and genus g.

Now, the *proof of Theorem A* will be done in 3 steps (Lemmas 2.8, 2.9 and 2.10) according to the value of n modulo 3.

Lemma 2.8: *If $n = 3k$, $k \in \mathbf{Z}$, $k \geq 3$ and $(d, g) \in \tilde{A}_k^n$ (see (1.15)), then there is a (smooth, irreducible) curve $C \subset X_k^n$, non-degenerate in $\mathbf{P}^n$, of degree d and genus g.*

Proof: 0) The subdomain of $\tilde{A}_k^n$ defined by $\alpha_{k+1}(d-1, n) < g \leq \alpha_k(d, n)$, $d \geq 2n + 1$ is covered by Theorem C.

1) **Claim:** *There exist (smooth, irreducible) curves $C \subset X_k^n$, non-degenerate in $\mathbf{P}^n$ of degree d and genus g, for*

$$g = [F_d^{k,n}(2(x_k(d, n) + 1))]_*, \ d \geq 2n + 1.$$

We check this by induction on $x_k(d, n) \geq 0$, using the obvious relations: $x_k(d+(n+k-1), n) = x_k(d, n)+1$, $F_{d+(n+k-1)}^{p,n}(r+2) = F_d^{k,n}(r)+(d+k-1)$, $p_a(\mathcal{D}+\mathcal{H}_k^n) = p_a(\mathcal{D})+(d+k-1)$ where $d = (\mathcal{D}\cdot\mathcal{H}_k^n)$, $\mathcal{H}_k^n = (k+2; k, 1^{3k-n+5})$. So, for $x_k(d, n) = 0$ take curves from the base-point free linear systems $\mathcal{D}_1 = [g + 2; g, 1^4, 0]$ or $\mathcal{D}_2 = [g + 2; g, 1^5]$ and then add succesively to $\mathcal{D}_1$ or $\mathcal{D}_2$ hyperplane sections $[\mathcal{H}_k^n]$.

2) **Claim:** *There exist (smooth, irreducible) curves $C \subset X_k^n$, non-degenerate in $\mathbf{P}^n$, of degree d and genus g, in the domain given by:*

$$F_d^{k,n}(2(x_k(d, n) + 1)) < g \leq B(d, n) = \pi_k(d, n), \ d \geq d_1(n).$$

(for $d_1(n)$, see (1.9)).

In order to prove the Claim it's enough to show that

$$d \geq d_1(n) \Rightarrow F_d^{k,n}(r_0(d, n) - 1) \leq F_d^{k,n}(2(x_k(d, n) + 1)). \tag{2.8}$$

Indeed, then the Claim follows from (2.6), Theorem 2.7, Lemma 2.6 (use that $[\max\{F_d^{k,n}(r) | \, r \in \mathbf{Z}\}]_* = \max\{[F_d^{k,n}(r)]_* | \, r \in \mathbf{Z}\}$) and the Claim from 2), using Lemma 2.4 (remark that $d_1(n) \geq d_0(n)$, see (2.7)).

In order to prove (2.8) write $d-k-1 = x_k(d, n)(4k-1)+\varepsilon$, $0 \leq \varepsilon \leq 4k-2$, so $2(x_k(d, n) + 1) = \dfrac{2(d + 3k - \varepsilon - 2)}{4k - 1}$. With $\rho_k(d, n)$ from (2.4) it's easy to see that

$$2(x_k(d, n) + 1) \leq \rho_k(d, n) \Leftrightarrow 2k - 1 \leq \varepsilon \leq 4k - 2 \tag{2.9}$$

$$2(x_k(d, n) + 1) > \rho_k(d, n) \Leftrightarrow 0 \leq \varepsilon \leq 2k - 2. \tag{2.10}$$

a) $2k-1 \le \varepsilon \le 4k-2$.

In this case, using the definition (2.5) of r_0, it follows that is necessary and sufficient (cf. (2.9)) that $2(x_k(d,n)+1)+1 \ge s(d,n)$, $(\forall)d \ge d_1(n)$. But $2x_k(d,n)+3 \ge \dfrac{2d+2k-1}{4k-1}$ (use that $\varepsilon \le 4k-2$) so it's enough to have

$$\frac{2d+2k-1}{4k-1} \ge s(d,n), \quad (\forall)d \ge d_1(n). \tag{2.11}$$

b) $0 \le \varepsilon \le 2k-2$.

We have $F_d^{k,n}(\beta) = F_d^{k,n}(2\rho_k(d,n)-\beta)$, $(\forall)\beta \in \mathbf{R}$. Taking $\beta = 2(x_k(d,n)+1)$, one obtains

$$F_d^{k,n}(2(x_k(d,n)+1)) = F_d^{k,n}(2(\rho_k(d,n)-x_k(d,n)-1)). \tag{2.12}$$

Moreover

$$\begin{cases} 2(\rho_k(d,n)-x_k(d,n)-1) \le \rho_k(d,n) \\ r_0-1 \le s(d,n) \le \rho_k(d,n), \ (\forall)d \ge d_1(n) \ (> d_0(n)). \end{cases}$$

From the definition (2.5) of r_0, we deduce that is enough to prove:

$$[2(\rho_k(d,n)-x_k(d,n)-1)]_* + 1 \ge s(d,n), \ (\forall)d \ge d_1(n). \tag{2.13}$$

We can check that

$$[2(\rho_k(d,n)-x_k(d,n)-1)]_* = \begin{cases} 2x_k(d,n), & 0 \le \varepsilon \le k-1 \\ 2x_k(d,n)+1, & k \le \varepsilon \le 2k-2. \end{cases} \tag{2.14}$$

If $k \le \varepsilon \le 2k-2$, from (2.13), (2.14), minorating $x_k(d,n)$ (use $\varepsilon \le 2k-2$), it follows that in order to prove (2.8) it is sufficient to prove

$$\frac{2d+2k}{4k-1} \ge s(d,n), \quad (\forall)d \ge d_1(n). \tag{2.15}$$

If $0 \le \varepsilon \le k-1$, we similarly show that's enough to prove

$$\frac{2d-1}{4k-1} \ge s(d,n), \quad (\forall)d \ge d_1(n). \tag{2.16}$$

Comparing (2.11), (2.15), (2.16) it follows that all three are true (and then (2.8) is true) as far as only (2.16) is true. But, replacing $s(d,n)$ from Lemma 2.4 c) we can see that the inequality

$$\frac{2d-1}{4k-1} \ge \frac{1}{3}(-4k+1+\sqrt{12d+16k(k+1)-38}), \ k \ge 3$$

is true exactly if $d \geq d_1(n)$, so (2.8) follows.

3) Using now Lemma 2.1 i), the assertion from our Lemma follows from 0), 1), 2).

* * *

Lemma 2.9: *If $n = 3k+1$, $k \in Z$, $k \geq 3$ and $(d,g) \in \tilde{A}_k^n$ (see (1.15)), then there is a (smooth, irreducible) curve $C \subset X_k^n$, non-degenerate in P^n, of degree d and genus g.*

Proof:

1) **Claim:** *There exist (smooth, irreducible) curves $C \subset X_k^n$, non-degenerate in P^n, of degree d and genus g in the domain given by:*

$$F_d^{k,n+1}(2(x_{k+1}(d-1,n)+1)) \leq g \leq F_d^{k,n}(2(x_{k+1}(d-1,n)+1)),\ d \geq 2n+1.$$

We will prove the Claim by constructing invertible sheaves $\mathcal{D} \in Pic(X_k^n)$ verifying the conditions (C1) - (C4) of Proposition 2.2 in the Gruson-Peskine coordinates $(d, r; \theta_1, \ldots, \theta_4)$, given by (2.2), (2.3), (C5) being automatically satisfied (Remark 2.3).

Denoting by $r := 2(x_{k+1}(d-1,n)+1)$, the domain from the Claim becomes

$$F_d^{k,n}(r) - \frac{1}{8}r^2 = F_d^{k,n+1}(r) \leq g \leq F_d^{k,n}(r),\ d \geq 2n+1. \tag{2.17}$$

We express $m := 2(F_d^{k,n}(r) - g) \in \mathbb{Z}$ as a sum of four squares $m = \sum_{i=1}^{4} \theta_i^2$, $\theta_i \in \mathbb{Z}$, $\theta_i \geq 0$, $i = \overline{1,4}$. Then $g = F_d^{k,n}(r) - \frac{1}{2}\sum_{i=1}^{4} \theta_i^2$ (see (2.3)). Then (C1) is satisfied, (C2) is also satisfied because $g \in \mathbb{Z}$ and (C3) follows using the domain of g (2.17) and ordering the θ_i. As for (C4):

$$-\theta_1 + \sum_{i=2}^{4} \theta_i \leq \sum_{i=2}^{4} \theta_i \leq 3\sqrt{\frac{\sum_{i=2}^{4} \theta_i^2}{3}} \leq 3\sqrt{\frac{\frac{r^2}{4} - \theta_1^2}{3}} \leq \frac{\sqrt{3}}{2} r.$$

It is enough to have: $\frac{\sqrt{3}}{2} r \leq d - (k+1)r$, i.e.

$$r \leq \frac{2d}{2k+2+\sqrt{3}}. \tag{2.18}$$

But $r = 2(x_{k+1}(d-1,n)+1) \leq \dfrac{2(d+3k-1)}{4k+1}$ so, in order to obtain (2.18), it's enough to have $\dfrac{2(d+3k-1)}{4k+1} \leq \dfrac{2d}{2k+2+\sqrt{3}}$, i.e.

$$d \geq 3k + \frac{7+6\sqrt{3}}{2} + \frac{21+11\sqrt{3}}{2(2k-1-\sqrt{3})}. \tag{2.19}$$

Because $d \geq 2n+1 = 6k+3$, it follows immediately that (2.19) is true for $k \geq 4$. As for $k = 3$ (i.e. $n = 10$), (2.19) is still true for $d \geq 24$. It remains $d \in \{21, 22, 23\}$; in this case we can choose $(\theta_1, \theta_2, \theta_3, \theta_4)$ from the set $\{(0,0,0,0), (0,0,0,1), (0,0,1,1), (0,1,1,1), (1,1,1,1)\}$ so that (C1) - (C4) hold.

2) **Claim:** *There exist (smooth, irreducible) curves $C \subset X_k^n$, non-degenerate in P^n, of degree d and genus g in the domain given by:*

$$F_d^{k,n}(2(x_{k+1}(d-1,n)+1)) < g \leq B(d,n) = \pi_k(d,n),\ d \geq d_1(n).$$

As in the Claim 2) from the proof of the previous Lemma (using the fact that we constructed curves of degree $d \geq 2n+1$ and genus $g = [F_d^{k,n}(2(x_{k+1}(d-1,n)+1))]_*$ in the Claim from 1)), in order to prove our Claim it's enough to show that:

$$d \geq d_1(n) \Rightarrow F_d^{k,n}(r_0(d,n)-1) \leq F_d^{k,n}(2(x_{k+1}(d-1,n)+1)). \tag{2.20}$$

We write $d-k-2 = x_{k+1}(d-1,n)(4k+1)+\varepsilon$, $0 \leq \varepsilon \leq 4k$. Doing computations we obtain

$$2(x_{k+1}(d-1,n)+1) \leq \rho_k(d,n) \Leftrightarrow \max\left(0, \frac{8k^2-k+1-d}{4k}\right) \leq \varepsilon \leq 4k \tag{2.21}$$

$$2(x_{k+1}(d-1,n)+1) > \rho_k(d,n) \Leftrightarrow 0 \leq \varepsilon \leq \frac{8k^2-k-d}{4k} (\text{if} d \leq 8k^2-k) \tag{2.22}$$

(for $\rho_k(d,n)$ see (2.4)).

a) $0 \leq \varepsilon \leq \dfrac{8k^2-k-d}{4k}$ (if $d \leq 8k^2-k$).

Then

$$F_d^{k,n}(2(x_{k+1}(d-1)+1)) = F_d^{k,n}(2(\rho_k(d,n) - x_{k+1}(d-1,n)-1)), \tag{2.23}$$

and

$$\begin{cases} 2(\rho_k(d,n) - x_{k+1}(d-1,n)-1) \leq \rho_k(d,n) \\ r_0 - 1 \leq s(d,n) \leq \rho_k(d,n),\ (\forall) d \geq d_1(n)\ (> d_0(n)) \end{cases}$$

So, in order to get (2.20), it's enough to prove that

$$[2(\rho_k(d,n) - x_{k+1}(d-1,n) - 1)]_* \geq s(d,n), \ (\forall)d \leq d_1(n). \tag{2.24}$$

But

$$[2(\rho_k(d,n) - x_{k+1}(d-1,n)]_* = \tag{2.25}$$

$$= \begin{cases} 2x_{k+1}(d-1,n), \ 0 \leq \varepsilon \leq \dfrac{4k^2-2k-d}{4k} \ (\text{if } d \leq 4k^2-2k) \\ 2x_{k+1}(d-1,n)+1, \max\left(0, \dfrac{4k^2-2k+1-d}{2k}\right) \leq \varepsilon \leq \dfrac{8k^2-k-d}{4k}. \end{cases}$$

Similar to the point b) of the proof of Claim 2) from the proof of previous lemma we see that (2.24) follows, if $0 \leq \varepsilon \leq \dfrac{4k^2-2k-d}{4k}$ (if $d \leq 4k^2-2k$), from

$$\frac{(4k+1)d-4k}{2k(4k+1)} \geq s(d,n), \ (\forall)d \geq d_1(n) \tag{2.26}$$

and, if $\max\left(0, \dfrac{4k^2-2k+1-d}{4k}\right) \leq \varepsilon \leq \dfrac{8k^2-k-d}{4k}$ (if $d \leq 8k^2-k$), from

$$\frac{(4k+1)d+4k^2-3k}{2k(4k+1)} \geq s(d,n), \ (\forall)d \geq d_1(n). \tag{2.27}$$

b) $\max\left(0, \dfrac{8k^2-k+1-d}{4k}\right) \leq \varepsilon \leq 4k.$

Doing similar computations as in the point a) of Claim 2) from the proof of previous lemma, we see that we need

$$\frac{2d+2k-1}{4k+1} \geq s(d,n) = -2(2k+1) + 2\sqrt{d+2(k-1)+(2k+1)^2}, \tag{2.28}$$

$$(\forall)d \geq d_1(n).$$

But, because $d \geq 2n+1$, we see that the inequality from (2.28) is true if and only if

$$d \geq d_1'(n) := \max(2n+1, -3k + \frac{1}{2}(4k+1)\sqrt{4k-5}). \tag{2.29}$$

So, in order to get (2.20) we need (2.26), (2.27), (2.28) to be true. We shall prove later that $d_1(n) \geq d_1'(n)$, $(\forall)n \geq 10$. It follows that all three

relations are true only if (2.26) is true, but this is true if and only if $d \geq d_1(n)$, so (2.20) is true (with $d_1(n)$ from (1.9)), q.e.d.

Finally, we shall prove that $d_1(n) \geq d_1'(n)$, $(\forall)n \geq 10$. First, it's easy to see that

$$4k^2 + 2k > d_1'(n), \quad (\forall)k \geq 3. \tag{2.30}$$

Secondly, let's suppose that $(\exists)n_0 \geq 10$, $n_0 = 3k_0+1$, such that $d_1(n_0) < d_1'(n_0)$. Then $(\exists)x \in \mathbf{Z}$ so that $d_1(n_0) < x < d_1'(n_0)$. Because $d_1(n)$ and $d_1'(n)$ are *minimal* so that (2.26), resp. (2.28) holds, it results that

$$\frac{2x + 2k_0 - 1}{4k_0 + 1} < s(x, n_0) < \frac{(4k_0 + 1)x - 4k_0}{2k_0(4k_0 + 1)}.$$

Without $s(x, n_0)$ we get an inequality, equivalent to $x > 4k^2 + 2k_0$. But $x < d_1'(n_0)$ so, without x, we obtain $d_1'(n_0) > 4k_0^2 + 2k_0$, which contradicts (2.30).

3) The lemma follows immediately from 1) and 2), using Lemma 2.1 ii).

* * *

Lemma 2.10: *If $n = 3k + 2$, $k \geq 2$ and $(d, g) \in \tilde{A}_k^n$, then there is a (smooth, irreducible) curve $C \subset X_k^n$, non-degenerate in $\mathbf{P}^n$ of degree d and genus g.*

Proof: 0) First, we show that we may *suppose* that $\delta_0(8) = 17$ (so, $\delta_0(n) = 2n + 1$, $(\forall)n \geq 8$, see Lemma 2.4 c)). For this, it's enough to construct curves of degree d and genus g on $X_2^8 \subset \mathbf{P}^8$ in the domain

$$\alpha_3(d-1, 8) = \begin{cases} 8, & d = 17 \\ 10, & d = 18 \\ 11, & d = 19 \\ 13, & d = 20 \\ 14, & d = 21 \end{cases} < g \leq \pi_2(d, 8) = \begin{cases} 10, & d = 17 \\ 12, & d = 18 \\ 13, & d = 19 \\ 15, & d = 20 \\ 17, & d = 21. \end{cases}$$

We give now the necessary list (specifying to the left the degree, in the middle the genus and to the right the linear system on X_2^8 containing the (smooth, irreducible) curve $C \subset X_2^8$, non-degenerate in $\mathbf{P}^8$, having the

previous degree and genus):

$d=17$	$g=9$	$[6;2,1,1,1]$
	$g=10$	$[7;3,2,2,1]$
$d=18$	$g=11$	$[7;3,2,1,1]$
	$g=12$	$[8;4,2,2,2]$
$d=19$	$g=12$	$[7;3,1,1,1]$
	$g=13$	$[8;4,2,2,1]$
$d=20$	$g=14$	$[8;4,2,1,1]$
	$g=15$	$[9;5,2,2,2]$
$d=21$	$g=15$	$[8;4,1,1,1]$
	$g=16$	$[9;5,2,2,2]$
	$g=17$	$[9;4,3,2,2]$

1) **Claim:** *There exist (smooth, irreducible) curves $C \subset X_k^n$, non-degenerate in P^n, of degree d and genus g, in the domain given by*

$$F_d^{k,n}(2x_{k+1}(d-1,n)-1) < g \leq B(d,n) = \pi_k(d,n)-\mu_k+\max(0,\varepsilon_k-3k-1),\ d \geq d_1(n).$$

We remark on the fact that curves on X_k^n of degree $d \geq 2n+1$ and genus $g = [F_d^{k,n}(2(x_k(d,n)+1))]_*$ can be constructed as in 1) from the proof of Lemma 2.8. We can see, writing $d = k+2+x_k(d,n)(4k+1)+\varepsilon$, $0 \leq \varepsilon \leq 4k$ that

$$2(x_k(d,n)+1) \leq \rho_k(d,n) \Leftrightarrow 2k \leq \varepsilon \leq 4k$$

$$2(x_k(d,n)+1) > \rho_k(d,n) \Leftrightarrow 0 \leq \varepsilon \leq 2k-1.$$

Using the equality

$$[2(\rho_k(d,n)-x_k(d,n)-1)]_* = \begin{cases} 2x_k(d,n),\ 0 \leq \varepsilon \leq k-1 \\ 2x_k(d,n)+1,\ k \leq \varepsilon \leq 2k-1 \end{cases}$$

and the obvious inequality $2x_k(d,n)+1 \geq 2x_{k+1}(d-1,n)+1$, it follows from (2.6), Theorem 2.7, Lemma 2.4 that is enough to show that

$$2x_{k+1}(d-1,n)+1 \geq s(d,n), \quad (\forall)d \geq d_1(n) \tag{2.31}$$

(for $s(d,n)$ see Lemma 2.4 e) and for $d_1(n)$ see (1.9)).

We remark that $2x_{k+1}(d-1,n)+1 \leq \rho_k(d,n)$, $(\forall)d \geq 2n+1$ and that $d_1(n) > d_0(n)$.

Because $2x_{k+1}(d-1,n)+1 \geq \dfrac{d-3k-2}{2k+1}$, it follows that it's enough to have

$$\frac{d-3k-2}{2k+1} \geq s(d,n) = -(8k+3)+\sqrt{(8k+3)^2+24k+8d-11}. \tag{2.32}$$

But this holds exactly for $d \geq d_1(n)$.

2) **Claim:** *There exist (smooth, irreducible) curves $C \subset X_k^n$, non-degenerate in $\boldsymbol{P}^n$, of degree d and genus g, in the domain from the (d,g)-plane given by the following conditions:*

$$F_d^{k,n+1}(2(x_{k+1}(d-1,n)+1)) \leq F_d^{k,n}(2x_{k+1}(d-1,n)-1);$$

$$[F_d^{k,n+1}(2(x_{k+1}(d-1,n)+1)),\ F_d^{k,n}(2x_{k+1}(d-1,n)-1)] \cap \mathbf{Z} \neq \emptyset;$$

$$F_d^{k,n+1}(2(x_{k+1}(d-1,n)+1)) \leq g \leq F_d^{k,n}(2x_{k+1}(d-1,n)-1),\ d \geq d_1(n).$$

We denote by $r := 2x_{k+1}(d-1,n)-1$. Then, it's easy to see that

$$d \geq d_1(n) \Rightarrow r \geq 3. \tag{2.33}$$

Let there be $m := 8(F_d^{k,n}(r)-g)$; then $m \equiv 3 \pmod 8$. From Gauss' theorem we deduce that $m = a_1^2+a_2^2+a_3^2$, $a_1,a_2,a_3 \in \mathbf{Z}$, a_1,a_2,a_3 odd numbers so that $|a_1| \leq a_2 \leq a_3$. Take $\theta_i := \dfrac{a_i}{2}$, $i=1,2,3$. Then $g = F_d^{k,n}(r) - \dfrac{1}{2}\sum_{i=1}^{3}\theta_i^2$.

From the bounds of the domain of the Claim it follows that

$$\sum_{i=1}^{3}\theta_i^2 \leq 2[F_d^{k,n}(r)-F_d^{k,n}(r+3)]+\frac{1}{4}(r+3)^2. \tag{2.34}$$

It can be checked that

$$F_d^{k,n}(r)-F_d^{k,n}(r+3) \leq -\frac{r}{2}. \tag{2.35}$$

We show now that $(d,r;\theta_1,\theta_2,\theta_3)$ verifies the conditions (C1) - (C4) from the Proposition 2.2 ((C5) is automatically satisfied, see Remark 2.3); this proves that the Claim is true.

(C1) is clear. For (C2) let's remark on the fact that (because a_1,a_2,a_3,r are odd numbers and $n=3k+2$)

$$E(\theta) := d+\frac{1}{2}(k-n+5)r-\theta_1-\theta_2-\theta_3 \in \mathbf{Z},\ (\forall)\theta \in \mathbf{Z}\left[\frac{1}{2}\right] \setminus \mathbf{Z}.$$

Moreover $E(\theta) - E(-\theta) \equiv 1 \pmod 2$; so, changing (if necessary) the sign of a_1, we obtain (C2). As for (C3), we remark on the fact that, using (2.34) and (2.35) we obtain

$$\theta_j \leq \sum_{i=1}^{3} \theta_i^2 \leq \left(\frac{r+1}{2}\right)^2 + 2, \ j = 1, 2, 3. \tag{2.36}$$

If $|\theta_j| > \frac{r}{2}$, from (C1) it follows that $|\theta_j| \geq \frac{r+2}{2}$, so $\theta_j^2 \geq \frac{(r+1)^2+9}{4}$ (using (2.33)). But $\frac{(r+1)^2+9}{4} > \left(\frac{r+1}{2}\right)^2 + 2$, which contradicts (2.36). So $|\theta_j| \leq \frac{r}{2}$, $j = 1, 2, 3$. Then, the conditions $|\theta_1| \leq \theta_2 \leq \theta_2$ follows from the similar conditions imposed to a_1, a_2, a_3. Finally, for (C4): $-\theta_1 + \theta_2 + \theta_3 = 3 \cdot \frac{-\theta_1 + \theta_2 + \theta_3}{3} \leq 3\sqrt{\frac{\theta_1^2 + \theta_2^2 + \theta_3^2}{3}} \leq$ (using (2.34), (2.35)) $\sqrt{3}\sqrt{\frac{(r+1)^2+8}{4}} \leq \frac{\sqrt{3}}{2}(r+2)$ (using (2.33) again). So, in order to get (C4) it's enough to gave $\frac{\sqrt{3}}{2}(r+2) \leq d - \frac{1}{2}(2k+3)r$, i.e. $r \leq \frac{2d - 2\sqrt{3}}{2k+2+\sqrt{3}}$. But this is true, because $r \leq \frac{d-3k-3}{2k+1}$.

3) • If $F_d^{k,n}(2x_{k+1}(d-1,n)-1) < F_d^{k,n+1}(2(x_{k+1}(d-1,n)+1))$, then there exist curves of genus g and degree $d \geq d_1(n)$ for $\alpha_{k+1}(d-1,n) < g \leq B(d,n)$ by 0), Claim 1) and Lemma 2.1 ii).

• If $F_d^{k,n+1}(2(x_{k+1}(d-1,n)+1)) \leq F_d^{k,n}(2x_{k+1}(d-1,n)-1)$, then there exist curves of degree $d \geq d_1(n)$ and genus g between $\alpha_{k+1}(d-1,n)$ and $F_d^{k,n}(2x_{k+1}(d-1,n)-1)$ by Claim 2) and Lemma 2.1 ii); use then Claim 2) to cover $\tilde{A}_k^n$.

* * *

Theorem A is now the union of Lemmas 2.8, 2.8 and 2.10.

* * *

3 The proof of Theorem B

The proof consists in a comparison between the *lower bound* of genus from our Theorem C and the *upper bound* of genus from Ciliberto-Sernesi Main Theorem from [CS] (for $\delta = 4$) and a filling of the distance between them when it exists.

We *defined* in [P 2] the following numerical functions (for $d, r \in \mathbf{Z}$, $\delta \in \{2,3,4\}$):

$$G_d^\delta(r,n) := r\left(d - nr + \frac{r-1}{2}\delta\right) \tag{3.1}$$

$$\begin{cases} y_\delta(d,n) := \left[\dfrac{d+n-\delta}{2n-\delta}\right]_* \\ \alpha''_{n-\delta+1}(d,n) := G_d^\delta(y_\delta(d,n)). \end{cases} \tag{3.2}$$

Lemma 3.1:

a) $G^\delta_{d+(2n-\delta)}(r+1) = G_d^\delta(r) + (d+n-\delta)$;

b) $y_\delta(d+(2n-\delta),n) = y_\delta(d,n)+1$;

c) $\alpha''_{n-\delta+1}(d+(2n-\delta),n) = \alpha''_{n-\delta+1}(d,n) + (d+n-\delta)$.

Proof: easy

We recall the definition of functions $\alpha_p(d,n)$ from (1.5) - (1.8). Then we have

Lemma 3.2 (Lemma 3.3.1 a) from [P 2]): *Let's denote by* $d' := d+(n+p-1)$. *Then:*

$x_p(d',n) = x_p(d,n)+1$; $t_p(d',n) = t_p(d,n)$; $u_p(d',n) = u_p(d,n)$;

$\alpha_p(d',n) = \alpha_p(d,n)+(d+p-1)$ ($\Leftrightarrow \alpha_p(d-(n+p-1),n) = \alpha_p(d,n)-d+n$).

Proof: easy.

Theorem 3.3 (see [P 1] for $\delta = 2$ and [CS] for $\delta \in \{3,4\}$): *Let there be* $\delta \in \{2,3,4\}$, $n \in Z$, $n \geq 2\delta - 1$, $d, g \in Z$, $d \geq 2n+1$ *and*

$$0 \leq g \leq \begin{cases} \alpha''_{n-\delta+1}(d,n), & \delta = 2 \\ \alpha''_{n-\delta+1}(d,n) - 1, & \delta \in \{3,4\}. \end{cases}$$

Then there is a (smooth, irreducible) non-degenerate curve $C \subset \boldsymbol{P}^n$ *of degree* d *and genus* g.

Proof: This theorem is, actually, proved in [P 1] ($\delta = 2$) and [CS] ($\delta \in \{3,4\}$) but the bound on the right is different. We show that we can consider the bound from Theorem 3.3. In [P 1] and [CS] the existence of curves of degree d and genus g in the range $0 \leq g < d-n$ is proved. These

curves are then succesively transformed r times obtaining curves having degrees d_r and genera g_r where (see [CS], p.311)

$$d = d_r - r(2n - \delta)$$

$$g = g_r - rd_r + \frac{r(r+1)}{2}(2n - \delta) - r(n - \delta).$$

The condition $0 \leq g < d - n$ becomes

$$G^{\delta}_{d_r}(r) \leq g_r < G^{\delta}_{d_r}(r+1)$$

(see (3.1)).

So, if $d, g \in \mathbf{Z}, d \geq n+1, g \geq 0$ and $r \in \mathbf{Z}$ so that

$$1 \leq r+1 \leq \begin{cases} y_\delta(d,n) & \text{, if } \frac{d+n-\delta}{2n-\delta} \notin \mathbf{Z} \\ y_\delta(d,n) - 1 & \text{, if } \frac{d+n-\delta}{2n-\delta} \in \mathbf{Z} \end{cases}$$

put $d_0 := d - r(2n-\delta), g_0 := g - rd + \frac{r(r+1)}{2}(2n-\delta) - r(n-\delta)$.Then $d_0 \geq n+1$. Let's denote by $\chi^\delta(d,n) := \frac{2d-\delta}{2(2n-\delta)}$ the point where the function (in r)$G^\delta_d(r)$ achieves its maximum. Because $\frac{d+n-\delta-1}{2n-\delta} - 1 \leq \chi^\delta(d,n)$ and $G^\delta_d(y_\delta(d,n)) = G^\delta_d(y_\delta(d,n) - 1)$ if $\frac{d+n-\delta}{2n-\delta} \in \mathbf{Z}$, we conclude that $\bigcup_{r as before} [G^\delta_d(r), G^\delta_d(r+1)] = [0, G^\delta_d(y_\delta(d,n))] = [0, \alpha''_{n-\delta+1}(d,n)]$. Since the existence of curves in the range $G^\delta_d(0) = 0 \leq g_0 < d_0 - n = G^\delta_d(1)$ is already proved , it results (using the previous transformations) that we obtained curves of degree d and genus g in the ranges $G^\delta_d(r) \leq g < G^\delta_d(r+1)$ for all r as before , so covering the domain from Theorem 3.3.

* * *

Lemma 3.4 (see [P 2], Prop. 3.2.2 i): *Let there be $\Sigma = \{P_0, P_1, \ldots, P_s\} \subset \mathbf{P}^2$ a set of general points and $S = Bl_\Sigma(\mathbf{P}^2)$. Let there be $\mathcal{D} = (p+2; p, 1^s) \in Pic(S)$. Then $h^0(\mathcal{D}) \neq \emptyset$, $[\mathcal{D}]$ has no base point and contains a (smooth, irreducible) curve for $s \leq 3p+3$.*

Proof: We specialize the points from Σ (except the last one) on a rational irreducible plane curve $\Delta_0 \subset \mathbf{P}^2$ of degree $p+1$ having only one ordinary singularity of multiplicity p : P_0 will be the singular point, $P_1, P_2, \ldots, P_{s-1} \in \Delta_0$ and $P_s \notin \Delta_0$. Let's denote by Δ the proper transformation of Δ in S. Let there be $D \in Div(S)$ so that $\mathcal{D} = \mathcal{O}_S(D)$. We get the following natural exact sequence of sheaves

$$0 \to \mathcal{O}_S(D - \Delta) \to \mathcal{O}_S(D) \to \mathcal{O}_\Delta(D) \to 0.$$

We consider the associated exact sequence

$$0 \to H^0(\mathcal{O}_S(D-\Delta)) \to H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_\Delta(D)) \to H^1(\mathcal{O}_S(D-\Delta)).$$

But $|D-\Delta| = [1;0,\ldots,0,1]$ so $h^1(\mathcal{O}_S(D-\Delta)) = 0$ and $|D-\Delta|$ has no base point; so $[\mathcal{D}] = [\mathcal{O}_S(D)]$ has no base point outside Δ. Moreover, we get the surjection $H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_\Delta(D)) \to 0$. So, $\mathcal{D} = \mathcal{O}_S(D)$ has no base point on Δ if $(D\cdot\Delta) \geq 2g(\Delta) = 0$, so for $s \leq 3p+3$ ($\Delta \in [p+2;p,1^{s-1},0]$). By a Bertini type theorem (see [Ha 2]), it results the conclusion of Lemma, because $h^0(\mathcal{O}_S(D)) = 3p+6-s \geq 3$ (use $h^1(\mathcal{D}) = 0$).

* * *

Lemma 3.5: *If $p,n \in Z$, $n \geq 3$ and $p \geq \dfrac{n-2}{3}$, let there be $\mathcal{H}_p^n = (p+2;p,1^{s_p^n}) \in Pic(X_p^n)$ and $\mathcal{D}_0 = (a+2;a,\nu_1,\nu_2,\ldots,\nu_{s_p^n}) \in Pic(X_p^n)$, $\nu_j \in \{0,1\}$, $j = \overline{1,s_p^n}$ and $\nu_j = 1$ for u values of the index $j (0 \leq u \leq s_p^n = 3p-n+5)$. Let there be*

$$\mathcal{D} := \mathcal{D}_0 + t\,\mathcal{H}_p^n.$$

Then, the condition

$$3a \geq u+n-7-(t+1)(n-4) \tag{3.3}$$

implies $[\mathcal{D}] \neq \emptyset$, $[\mathcal{D}]$ without base points and contains a (smooth, irreducible) curve.

Proof: Write $\mathcal{D} = t\mathcal{D}_x + \mathcal{R}$ where $\mathcal{D}_x = (x+2;x,1^{s_p^n})$, $\mathcal{R} = (b+2;b,\nu_1,\ldots,\nu_{s_p^n})$, $b = a+tp-tx$. Take now $x \in \mathbf{Z}$, $x \geq 0$ *minimal* so that Lemma 3.4 applies. Then

$$3p-n+2 \leq 3x \leq 3p-n+4. \tag{3.4}$$

In order to obtain the conclusion from the Lemma for $[\mathcal{D}]$, it follows that's enough to have the same for $[\mathcal{R}]$ so, by Lemma 3.4, we need

$$u \leq 3b+3, \quad b = a+tp-tx. \tag{3.5}$$

Replacing b, (3.5) becomes $u \leq 3a+3tp-3tx+3$, or $t(3x) \leq 3a+3tp+3-u$; using (3.4), we can see that this last inequality is implied by $t(3p-n+4) \leq 3a+3tp+3-u$, which is equivalent with (3.3).

* * *

Remark 3.6: *Lemma 3.5 is a particular case of Lemma 3.2.4 from [P 2]; here we need only this particular case.*

In order to prove Theorem B we need two more lemmas:

Lemma 3.7: *Let there be $d, g \in Z$, $d \geq 2$, $n \in Z$, $n \geq 3$ and $\alpha''_{n-3}(d,n) \leq g \leq \alpha_{n-3}(d,n)$ (see (3.2) and (1.5)). Then there is $\mathcal{D} \in Pic(X^n_{n-3})$ (see (2.1)) so that $(\deg \mathcal{D}, p_a(\mathcal{D})) = (d, g)$ (here $\deg \mathcal{D} = (\mathcal{D} \cdot \mathcal{H}^n_{n-3})$).*

Remark 3.8: $\alpha''_{n-3}(d,n) \leq \alpha_{n-3}(d,n) \Leftrightarrow d \neq 2 + t(2n-4)$, $t \in \mathbf{Z}$.

Proof (of Lemma 3.7): We construct the necessary invertible sheaves by induction on $x_{n-3}(d,n) \geq 0$, using the numerical properties of the functions α''_{n-3} and α_{n-3} from Lemmas 3.1 c) ($\delta = 4$) and 3.2 ($p = n-3$).

If $x_{n-3}(d,n) = 0$ (i.e. $2 \leq d \leq 2n-3$), we take $\mathcal{D} = \mathcal{D}_0 = (a+2; a, 1^u, 0^{2n-4-u}) \in Pic(X^n_{n-3})$ where $u \in \{2n-d-2, 2n-d-1, \ldots, 2n-5\}$ if $3 \leq d \leq n-1$ and $u \in \{d-2, d-1, \ldots, 2n-5\}$ if $n \leq d \leq 2n-3$; here $d = \deg \mathcal{D}_0 = (\mathcal{D}_0 \cdot \mathcal{H}^n_{n-3})$, where $\mathcal{H}^n_{n-3} = (n-1; n-3, 1^{2n-4}) \in Pic(X^n_{n-3})$ is the hyperplane section sheaf. In order to cover the whole domain of lemma, we consider then the invertible sheaves

$$\mathcal{D} = \mathcal{D}_0 + t\mathcal{H}^n_{n-3} \in Pic(X^n_{n-3}), \quad t \geq 0.$$

Remark 3.9: $\alpha''_{n-3}(d,n) \leq \alpha_{n-3}(d,n) - 1 \Leftrightarrow d \neq \tau + t(2n-4)$, $\tau \in \{0,1,2,3,4\}$, $t \in \mathbf{Z}$.

Remark 3.10: *The invertible sheaves used in the proof of Lemma 3.7 covering only the domain from Remark 3.9 are of the following form:*

$$\mathcal{D} = \mathcal{D}_0 + t\mathcal{H}^n_{n-3},\ t \in Z,\ t \geq 0,\ \mathcal{D} \in Pic(X^n_{n-3})$$
$$\mathcal{D}_0 = (a+2; a, 1^u, 0^{2n-4-u}) \in Pic(X^n_{n-3}),$$

where

$$a = \frac{d_0 - 2n + u + 2}{2},\ a \in \mathbf{Z}$$
$$d_0 \in \{5, 6, \ldots, 2n-5\} \quad \text{and}$$
$$u \in \{2n - d_0 - 2, 2n - d_0 - 1, \ldots, 2n-7\} \text{ if } \quad 5 \leq d_0 \leq n-1$$
$$u \in \{d_0 - 2, d_0 - 1, \ldots, 2n-7\} \text{ if } n \leq d_0 \leq 2n-5.$$
$$(d_0 = \deg \mathcal{D}_0 = (\mathcal{D}_0 \cdot \mathcal{H}^n_{n-3}).$$

Lemma 3.11: *Let there be $d, g \in Z$, $n \in Z$, $n \geq 7$ and $\alpha''_{n-3}(d,n) \leq g \leq \alpha_{n-3}(d,n) - 1$. Then:*

i) if $d \geq 3n-5$ there is a (smooth, irreducible) curve $C \subset X^n_{n-3}$, non-degenerate in $\mathbf{P}^n$, of degree d and genus g;

ii) if $2n+1 \leq d \leq 3n-6$ there is a (smooth, irreducible) curve $C \subset X_p^n$, where $p = \left[\frac{2n-5}{3}\right]_*$ non-degenerate in $\mathbf{P}^n$, of degree d and genus g.

Proof: i) 1) If $d \geq 4n-6$, $d \neq \tau + t(2n-4)$, $\tau \in \{2,3,4,2n-4,2n-3\}$ (see Remark 3.9), $t \geq 2$, $t \in \mathbf{Z}$ we use for the sheaves $\mathcal{D} = \mathcal{D}_0 + t\mathcal{H}_{n-3}^n$, $t \geq 2$, used in the proof of Lemma 3.7 (see Remark 3.10) the criterion given by the relation (3.3) from Lemma 3.5. For $t \geq 2$ this becomes

$$3a \geq u - 2n + 5. \tag{3.6}$$

Replacing a from the Remark 3.10, (3.6) becomes

$$3d_0 + u \geq 2n + 4. \tag{3.7}$$

Minorating u and, after that, minorating d_0 in the two situations with respect to d_0 from Remark 3.10, we can see that (3.7) (hence (3.6) also) is satisfied.

2) Apply now again (3.3) from Lemma 3.5 for $t = 1$. As before, we get smooth curves for $3n-5 \leq d \leq 4n-7$, $d \neq \tau + (2n-4)$, $\tau \in \{2n-4, 2n-3\}$ (see Remark 3.9). These curves are non-degenerate in $\mathbf{P}^n$ because they have degree $\geq 3n-5 > 2n-4 = \deg X_{n-3}^n$.

From 1) and 2) it follows that i) is proved.

ii) Because $2n+1 \leq d \leq 3n-6$, we have $\alpha''_{n-3}(d,n) = d-n$ and $\alpha_{n-3}(d,n) = \left[\frac{3d-4n+1}{2}\right]_*$. So, we need to cover the domain

$$d - n \leq g \leq \left[\frac{3d-4n+1}{2}\right]_*, \quad 2n+1 \leq d \leq 2n-6. \tag{3.8}$$

Now, let there be (d,g) in the domain (3.8). Put

$$p := \left[\frac{2n-5}{3}\right]_* \tag{3.9}$$

$$\begin{cases} d_0 := d - (n+p-1) \\ g_0 := g - (d_0 + p - 1) = g - d + n \end{cases} \tag{3.10}$$

$$u := 2g_0 - d_0 + 2p + 4 = 2g - 3d + 3p + 3n + 3. \tag{3.11}$$

Then, we can check that:

$$0 \leq u \leq s_p^n = 3p - n + 5. \tag{3.12}$$

Put $\mathcal{D}_0 := (g_0+2; g_0, 1^u, 0^{s_p^n-u}) \in Pic(X_p^n)$, with p, g_0, u from (3.9) - (3.11) (use (3.12) also). Then $p_a(\mathcal{D}_0) = g_0$ and $\deg(\mathcal{D}_0) = (\mathcal{D}_0 \cdot \mathcal{H}_p^n) = d_0$ (from (3.10)). Put now $\mathcal{D} := \mathcal{D}_0 + \mathcal{H}_p^n \in Pic(X_p^n)$; $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n})$. Then $p_a(\mathcal{D}) = g_0 + (d_0 + p - 1) = g$, $\deg(\mathcal{D}) = d_0 + (n+p-1) = d$ (use (3.10)). We apply now Lemma 3.5 (the criterion (3.3)) for $\mathcal{D}$ ($t = 1$). It is easy to see that this works, so we get curves, non-degenerate by degree reasons. So ii) is proved.

* * *

Theorem B follows now from Theorem 3.3 and Lemma 3.11.

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