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A NONLINEAR EVOLUTION EQUATION IN AN ORDERED SPACE, ARISING FROM KINETIC THEORY

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ABSTRACT. We investigate the Cauchy problem for a nonlinear evolution equation, formulated in an abstract Lebesgue space, as a generalization of various Boltzmann kinetic models. Our main result provides sufficient conditions for the existence, uniqueness, and positivity of global in time solutions. The proof is based on ideas behind a well-known monotonicity method, originally developed within the existence theory of the classical Boltzmann equation in L^1. Our application examples concern Smoluchowski's coagulation equation, a Povzner-like equation with dissipative collisions, and a Boltzmann model with chemical reactions.

1. Introduction and formulation of the problem

In a couple of well-known papers in kinetic theory, Arkeryd [1] introduced a monotonicity method (see also [2]) to solve the full initial value problem (i.v.p.) for the space-homogeneous Boltzmann equation in L^1. To this end, a priori estimates (mass and kinetic energy conservation) were cleverly used to replace the original Boltzmann equation by an equivalent one, suitable for monotone iteration with respect to the natural order of L^1-real. To handle Boltzmann collision operators with unbounded integral kernels, Arkeryd introduced monotone sequences of collision-like operators with bounded kernels. This resulted into a convergent approximation scheme, applicable to operators satisfying the so-called Povzner inequality [1], [3]. Arkeryd proved the convergence of the scheme, by taking advantage of the monotone completeness of L^1, and applying conveniently the additivity of the L^1-norm on the positive cone in L^1.

The above line of reasoning has proved applicable to other Boltzmann-like equations (see, e.g., [4]-[8]). However, in the absence of general results, extending Arkeryd's monotonicity scheme to other models has not been always straightforward. Indeed, each model has actually required a rather specific analysis, where the construction of a suitable operator approximation, like in Arkeryd's original argument, seems to be a key issue.

Under these circumstances it is tempting to develop the ideas behind Arkeryd's method within a more general framework, in view of further possible applications.

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Given the above motivation, in this paper we consider the i.v.p.

$$\frac{df}{dt} = Q(t,f) = Q^+(t,f) - Q^-(t,f), \quad f(0) = f_0 \in X_+ \quad (t > 0), \tag{1.1}$$

formulated in a separable Banach lattice X, with positive cone X_+.

Specifically, X is an abstract Lebesgue (AL) space, i.e., a Banach lattice whose norm satisfies

$$\|g + h\| = \|g\| + \|h\| \quad (g, h \in X_+). \tag{1.2}$$

(We refer to Section 2 for useful terminology and facts from Banach lattice theory.)

In Eq. (1.1), Q^+ and Q^- are mappings defined from $\mathbb{R}_+ \times \mathcal{D}$ to X, for some $\mathcal{D} \subset X$ such that $\mathcal{D} \cap X_+$ is dense in X_+ (we adopt the convention $\mathbb{R}_+ := [0, \infty)$).

The following properties are assumed for $Q^\pm$:

a) For a.e. $t \geq 0$, the operators $Q^\pm(t, \cdot) : \mathcal{D} \mapsto X$ are positive (i.e., map $\mathcal{D} \cap X_+$ into X_+) and isotone (i.e., are order-preserving mappings).

b) The mappings $\mathbb{R}_+ \ni t \mapsto Q^\pm(t, g(t)) \in X_+$ are measurable for any Lebesgue measurable function $g : \mathbb{R}_+ \mapsto X$ that satisfies $g(t) \in \mathcal{D} \cap X_+$ a.e. on $\mathbb{R}_+$.

An important special case of the above setting is the autonomous problem (i.e., the terms $Q^\pm(t,f) = Q^\pm(f)$ do not depend explicitly on time).

We recall that a function $f : \mathbb{R}_+ \mapsto X$ is a strong solution of Eq. (1.1), if it is absolutely continuous on $\mathbb{R}_+$, differentiable a.e. on $\mathbb{R}_+$, satisfies Eq. (1.1) a.e. on $\mathbb{R}_+$, and verifies the initial condition. Equivalently, f is a strong solution of Problem (1.1) if it is solution of the integral equation

$$f(t) = f_0 + \int_0^t Q(s, f(s))ds \quad (t \geq 0), \tag{1.3}$$

where the integral is in the sense of Bochner.

We are interested in the existence and uniqueness of positive (i.e., in X_+) strong solutions of Eq. (1.1), under additional hypotheses ((A_0)-(A_3) in Section 3) which abstract further properties of the Boltzmann model considered in [1], and enables us to extend Arkeryd's monotonicity techniques to our setting.

More precisely, as will be seen later on, assumptions (A_0)-(A_3) guarantee some dissipation (conservation) properties for Eq. (1.1) in the following sense: There exists a positive, densely defined, closed linear operator $\Lambda : \mathcal{D}(\Lambda) \subset X \mapsto X$ such that, for any sufficiently regular positive solution $f(t) \in \mathcal{D}(\Lambda^2)$ of Eq. (1.1), the quantity $\|\Lambda f(t)\|$ is dissipated (conserved), i.e., is decreasing (constant) in t, and $\|\Lambda^2 f(t)\|$ is locally bounded in t. In particular,

$$\|\Lambda f(t)\| \leq \|\Lambda f_0\| \quad (t \geq 0). \tag{1.4}$$

The law of decrease of $\|\Lambda f(t)\|$ (formula (3.5) in Section 3) generalizes a priori estimates introduced in [1]. In determining the behaviour of $\|\Lambda^2 f(t)\|$, a major role appears to be played by an abstract version (formula (3.6) in Section 3) of the Povzner inequality [1], [3].

We are also interested in the following problem in X, related to Eq. (1.1)

$$\frac{df}{dt} = Af + Q(t,f), \quad f(0) = f_0 \in X_+ \quad (t > 0), \tag{1.5}$$

with Q as in Eq. (1.1). Here A is the infinitesimal generator of a C_0 group of positive linear isometries on X, which commutes with Λ.

Our main result, Theorem 3.1 formulated in Section 3, provides sufficient conditions for the existence and uniqueness of positive, strong solutions to problem

(1.1). In the same section, Proposition 3.1 yields information on the dissipation properties of Eq. (1.1) and additional moment-like estimations . Finally, Corollary 3.1 extends the results of Theorem 3.1 to the case of mild solutions (cf. Section 3) of problem (1.5).

The proofs of Theorem 3.1 and Proposition 3.1 are detailed in Section 4. This is achieved within an abstract analysis, along the monotonicity ideas of [1], but without appealing to any operator approximation as in Arkeryd's original argument.

Thus, the results obtained in [1] for the autonomous, conservative example of the Boltzmann equation are extended to a more general, abstract framework. The latter includes both autonomous and non-autonomous equations, as well as models with dissipation (conservation) properties, in the sense discussed before.

Various models of the classical kinetic theory can be regarded as particularized versions of Eqs. (1.1) and (1.5). In this respect, besides the space-homogeneous Boltzmann equation considered in [1], and the Boltzmann-like models discussed in [4]-[8], one can also mention the Povzner equation [3], the so-called generalized Boltzmann equation [9]. and the Smoluchowski's coagulation equation [10]-[11]. Other illustrations may come from quantum kinetics. Also, some kinetic models of applied science [12] can be related to (1.1) and (1.5).

In the above examples. Q^+ and Q^- are usually the so-called gain (creation) and loss (destruction) nonlinear operators of the considered model, respectively. In general, Λ is related to physical quantities specific to each model. Formula (1.4), with inequality (equality) sign, implies the dissipation (conservation) of those quantities. Finally, in some cases, A is the so-called free-streaming operator.

Section 5 is devoted to three basic applications: the continuous Smoluchowski's coagulation equation, a space-dependent Povzner-like model with dissipative collisions [7], and a generalized Boltzmann model with chemical reactions [6]. We obtain improved statements and simplified proofs to some known results, as well as a unified view-point on the existence theory of global in time solutions. In particular, Theorem 5.1 provides the existence of strong solutions to the continuous Smoluchowski's coagulation model with general initial data, including both the case of finite and infinite initial mass. Theorem 5.1 seems also to bring some novelty, as it proves the uniqueness of the above solutions under rather general assumptions on the coagulation kernel (see (5.4) in Section 5). Similar results can be stated for the discrete Smoluchowski coagulation equation. Further, the results obtained for the space-dependent Boltzmann model with dissipative collisions, can be easily transcribed to the space-homogeneous version of the model (which can be regarded also as an equation for granular flows [13], [14]). The last application of Section 5 proves a more general version of an earlier result [6] on the Boltzmann equation with chemical reactions.

Although our work refers only to the Cauchy problem, mixed problems can be also considered, under suitable monotonicity conditions.

2. Preliminaries

We begin with some terminology and facts related to Banach lattices [15], [16].

As mentioned in Introduction, the frame of our analysis is a separable AL-space X with norm $\|\cdot\|$, order $\leq$, and positive cone X_+. Related to the order of X, we shall also use the standard notations $(g \geq h) \Leftrightarrow (h \leq g)$, as well as $(g < h) \Leftrightarrow (h > g) \Leftrightarrow (g \leq h$ and $g \neq h)$. In addition, for any $g \in X$, we denote $|g| := g_+ + g_-$, where $g_+ := g \vee 0$ and $g_- := (-g) \vee 0$. Examples of AL-spaces are L^1-real and the real subspace of self-adjoint trace-class operators (with trace norm). Actually, according to Kakutani's theorem (see, e.g., [15]), every AL-space is isometrically isomorphic (as an ordered vector space) to a space of type L^1. AL-spaces are *monotone complete*, in the sense that any increasing (i.e., directed $\leq$) norm-bounded family converges. The norm of an AL-space is *order continuous*, i.e., any directed $\geq$ filters that converges to 0 is also norm convergent to 0. A map $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$, with $\mathcal{D}(\Gamma) \cap X_+ \neq \emptyset$, is called *positive (strictly positive)* if $0 \leq \Gamma g$ for $0 \leq g \in \mathcal{D}(\Gamma)$ (if $0 < \Gamma g$ for $0 < g \in \mathcal{D}(\Gamma)$). Further, $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is called *isotone (strictly isotone)* if $\Gamma g \leq \Gamma h$, whenever $g \leq h$ (if $\Gamma g < \Gamma h$, whenever $g < h$), $g, h \in \mathcal{D}(\Gamma)$. Obviously, if $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is isotone, $0 \in \mathcal{D}(\Gamma)$ and $0 \leq \Gamma(0)$, then Γ is positive.

Here, we introduce two more definitions:

A subset $\mathcal{M} \subset X$ is called *p-saturated* (positively saturated) if $\mathcal{M} \cap X_+ \neq \emptyset$, and from $0 \leq g \leq h \in \mathcal{M}$, it follows that $g \in \mathcal{M}$.

An operator $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is *o-closed* (closed with respect to the order) if for any increasing sequence $\{g_n\} \subset \mathcal{D}(\Gamma)$ such that $\{g_n\}$ is increasing and convergent (in symbols, $\nearrow$) to some g, and $\{\Gamma g_n\}$ is Cauchy, one has $g \in \mathcal{D}(\Gamma)$ and $\lim_{n\to\infty} \Gamma g_n = \Gamma g$. Clearly, any closed mapping is also o-closed.

Concerning the integration of X-valued real functions, we recall the following property (see, e.g., [17]): Let $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ be a closed linear operator. If h is a Bochner integrable function defined on some measurable set $\mathbb{S} \in \mathbb{R}$, with values in $\mathcal{D}(\Gamma)$, and Γh is Bochner integrable, then

$$\Gamma \int_{\mathbb{S}} h(s)ds = \int_{\mathbb{S}} \Gamma h(s)ds. \tag{2.1}$$

We further note that, as X is an AL-space, if $h : \mathbb{R} \mapsto X_+$ is Bochner integrable, then property (1.2) gives

$$\left\| \int_{S} h(s)ds \right\| = \int_{S} \|h(s)\| \, ds \tag{2.2}$$

for any measurable set S of $\mathbb{R}$, the integral being in the sense of Lebesgue's integral.

Next, we recall that a *positive C_0 semigroup* on X is a C_0 semigroup of positive linear operators on X. If $\{S^t\}_{t\geq 0}$ is a positive C_0 semigroup on X, then its infinitesimal generator G is densely defined and closed (as the infinitesimal generator of a C_0 semigroup). Moreover, G^k is densely defined and closed, $k = 2, 3, \ldots$.

Additional properties are stated in the following lemma.

Let I denote the identity on X. Set $\mathcal{D}_+^\infty(G) := \cap_{k=1}^\infty \mathcal{D}(G^k) \cap X_+$.

Lemma 2.1. *a) The sets $\mathcal{D}(G^k) \cap X_+$, $k = 1, 2, \ldots$, and $\mathcal{D}_+^\infty(G)$ are dense in X_+.*

b) Suppose that there is some number $\gamma \geq 0$ such that

$$(G + \gamma I)g \leq 0 \quad (g \in \mathcal{D}(G) \cap X_+). \tag{2.3}$$

Then $\mathcal{D}(G^k) \cap X_+$, $k = 1, 2, \ldots$, and $\mathcal{D}_+^\infty(G)$ are p-saturated. Moreover, for any $h \in X_+$,

$$0 \leq S^t h \leq \exp(-\gamma t)h \quad (t \geq 0), \tag{2.4}$$

and one can find an increasing sequence $\{h_n\} \subset \mathcal{D}_+^\infty$, such that $h_n \nearrow h$ as $n \to \infty$.

Proof. a) We simply apply Gelfand's construction (see [17]: p. 308), and approximate any $h \in X_+$, by a sequence $\widetilde{h}_n \to h$ as $n \to \infty$, where

$$\mathcal{D}_+^\infty(G) \ni \widetilde{h}_n := n \int_0^\infty \varphi(nt) S^t h dt \quad (n = 1, 2, ...) \tag{2.5}$$

for some $\varphi \in C_0^\infty(0, \infty; \mathbb{R}_+)$ satisfying

$$\int_0^\infty \varphi(t) dt = 1.$$

b) We show by induction that

Let $0 \leq g \leq h \in \mathcal{D}(G) \cap X_+$ and $n \in \mathbb{N}$ sufficiently large. Recall that $\{S^t\}_{t \geq 0}$ is positive, and observe that $(-G)$ is positive, by (2.3). Then clearly,

$$g_n := n \int_0^\infty \exp(-nt) S^t g dt = n(nI - G)^{-1} g \in \mathcal{D}(G) \cap X_+, \tag{2.6}$$

and the sequence $\{g_n\}$ is increasing. Consequently, $\{-Gg_n\}$ is also positive and increasing. Since $\{-Gg_n\}$ is norm-bounded by $\|Gh\|$, it follows that $\{Gg_n\}$ is convergent, by the monotone completeness of X. On the other hand, (2.6) implies $\lim_{n \to \infty} g_n = g$, hence $g \in \mathcal{D}(G)$, because G is closed.

Now a straightforward induction, applying the positivity of $(-G)$, shows that $\mathcal{D}(G^k)$ is p-saturated for all k (hence, $\mathcal{D}_+^\infty(G)$ is also p-saturated).

To prove (2.4), note that if the number $\mu > 0$ is sufficiently large, then by (2.3),

$$(\mu I - G)^{-1} h \leq (\mu + \gamma)^{-1} h \quad (h \in X_+). \tag{2.7}$$

Now we get (2.4), applying (2.7) in the formula (see [17]: p. 354, Relation (E_9))

$$S^t h = \lim_{k \to \infty} [\frac{k}{t}(\frac{k}{t} I - G)^{-1}]^k h, \quad (h \in X, \quad t > 0).$$

To construct a sequence $\{h_n\}$ as stated in b), we apply a standard argument of convergence with regulator: Starting with $\left\{\widetilde{h}_n\right\} \subset \mathcal{D}_+^\infty$ as in (2.5), we choose a subsequence $\left\{\widetilde{h}_{n_i}\right\}$ such that $\left\|h - \widetilde{h}_{n_i}\right\| \leq 2^{-i}$, $i = 1, 2, ...$ Next define $h_n := \inf_{i \geq n} \widetilde{h}_{n_i}$. Then (2.4) implies $0 \leq h_n \leq \widetilde{h}_n \leq h$ for all $n = 1, 2, ...$ Evidently, the sequence $\{h_n\}$ is increasing and norm-bounded by $\|h\|$. Then by the monotone completeness of X, there is $h' \in X$ such that $h_n \nearrow h'$ as $n \to \infty$. Since $\mathcal{D}_+^\infty$ is p-saturated and $\left\{\widetilde{h}_{n_i}\right\} \subset \mathcal{D}_+^\infty$, it follows that also $\{h_n\} \subset \mathcal{D}_+^\infty$. To conclude the lemma, it is sufficient to prove that $h' = h$. Since clearly $h' \leq h$, we need only show that $h' \geq h$. To this end, observe that $z := \sum_i i \left|h - \widetilde{h}_{n_i}\right| < \infty$, because $\left\|h - \widetilde{h}_{n_i}\right\| \leq 2^{-i}$. Consequently, for any number $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that if $i \geq N_\varepsilon$, then $h - \widetilde{h}_{n_i} \leq \varepsilon z$. Therefore, $h' \geq h - \varepsilon z$ for any $\varepsilon > 0$, so that $h' \geq h$. □

Finally, related to Eq. (1.1) we consider the following "dissipativity" property. Let $\mathcal{M}$ be a subset of $\mathcal{D} \cap X_+$ dense in X_+.

Definition 2.1. *A closed positive linear operator $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is called of type D on $\mathcal{M}$ (with respect to Eq. (1.1)) if $\mathcal{M} \subset \mathcal{D}(\Gamma)$, $Q^\pm(t, \mathcal{M}) \subset \mathcal{D}(\Gamma)$ a.e. on $\mathbb{R}_+$, and for any $g \in \mathcal{M}$,*

$$0 \leq \Delta(t, g; \Gamma, Q) := \left\|\Gamma Q^-(t, g)\right\| - \left\|\Gamma Q^+(t, g)\right\| \quad (t \geq 0 \quad a.e.). \tag{2.8}$$

If Γ is of type D on $\mathcal{M}$, then the following property can be easily established:

Lemma 2.2. *Suppose that* g_0, $g(t)$, $h(t) \in \mathcal{M}$, $t \geq 0$ *a.e., with* $Q^{\pm}(\cdot, h(\cdot))$, $\Gamma Q^{\pm}(\cdot, h(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$, *and*

$$g(t) \leq g_0 + \int_0^t Q(s, h(s))ds \quad (t \geq 0). \tag{2.9}$$

Then

$$\|\Gamma g(t)\| + \int_0^t \Delta(s, h(s); \Gamma, Q)ds \leq \|\Gamma g_0\| \quad (t \geq 0). \tag{2.10}$$

Moreover, (2.10) holds with equality sign for any $t \geq 0$, *provided that there is equality in (2.9) for all* $t \geq 0$.

Proof. As Γ is positive, obviously, $\Gamma g(t), \Gamma g_0, \Gamma Q^{\pm}(t, h(t)) \in X_+$. Hence, applying Γ to (2.9) and using (2.1), we obtain

$$0 \leq \Gamma g(t) + \int_0^t \Gamma Q^-(s, h(s))ds \leq \Gamma g_0 + \int_0^t \Gamma Q^+(s, h(s))ds \quad (t \geq 0). \tag{2.11}$$

Then, by property (1.2),

$$\|\Gamma g(t)\| + \left\| \int_0^t \Gamma Q^-(s, h(s))ds \right\| \leq \|\Gamma g_0\| + \left\| \int_0^t \Gamma Q^+(s, h(s))ds \right\|,$$

which implies immediately (2.10), by virtue of (2.2) and (2.8).

A similar argument shows that if there is equality in (2.9), for all $t \geq 0$, then (2.10) also holds true with equality sign for any $t \geq 0$. □

3. Main result

We first complete the setting of Eq. (1.1) with additional hypotheses. In particular, we introduce in a precise manner the operator Λ, mentioned in the first section.

Specifically, we assume that there exists a linear operator $\Lambda : \mathcal{D}(\Lambda) \subset X \mapsto X$, with $\mathcal{D}(\Lambda) \subset \mathcal{D}$ and $Q^{\pm}(t, \mathcal{D}(\Lambda^k) \cap X_+) \subset \mathcal{D}(\Lambda^{k-1})$, $t \geq 0$ a.e., $k = 2, 3$, such that:

(A_0) The operator $(-\Lambda)$ is the infinitesimal generator of a C_0 semigroup of positive linear operators on X, and there is a number $\lambda_0 > 0$ such that

$$(\Lambda - \lambda_0 I)g \geq 0 \quad (g \in \mathcal{D}(\Lambda) \cap X_+). \tag{3.1}$$

(A_1) For a.e. $t \geq 0$,

$$\Delta(t, g) := \Delta(t, g; \Lambda, Q) \geq 0 \quad (g \in \mathcal{D}(\Lambda^2) \cap X_+), \tag{3.2}$$

and the map $\mathcal{D}(\Lambda^2) \cap X_+ \ni g \mapsto \Delta(t, g) \in \mathbb{R}_+$ is isotone.

(A_2) There exists a non-decreasing convex function $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$a(\|\Lambda g\|)\Lambda g - Q^-(t, g) \geq 0, \quad (g \in \mathcal{D}(\Lambda) \cap X_+, \quad t \geq a.e.), \tag{3.3}$$

and for a.e. $t \geq 0$, the map $\mathcal{D}(\Lambda) \cap X_+ \ni g \mapsto a(\|\Lambda g\|)\Lambda g - Q^-(t, g) \in X$ is isotone.

(A_3) There exists a non-decreasing function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}_+$, and there is an operator $\Lambda_1 : \mathcal{D}(\Lambda_1) \subset X \mapsto X$ of type D on $\mathcal{D}(\Lambda^2) \cap X_+$ such that

$$-\Delta(t, g; \Lambda^2, Q) \leq \rho(\|\Lambda_1 g\|) \left\|\Lambda^2 g\right\| \quad (g \in \mathcal{D}(\Lambda^3) \cap X_+, \, t \geq 0 \text{ a.e.}). \tag{3.4}$$

We assume that for a.e. $t \geq 0$, the operators $Q^{\pm}(t,\cdot)$ are o-closed and their common domain $\mathcal{D}$ is p-saturated.

At this point, some remarks are in order.

By Lemma 2.1a) and assumption (A_0), it follows that $\mathcal{D}(\Lambda^k) \cap X_+$, $k = 1,2,...$, and $\mathcal{D}_+^\infty := \mathcal{D}_+^\infty(\Lambda)$ are p-saturated and dense in X_+. Obviously, (3.1) shows that Λ is positive. Thus invoking (3.2), we get that Λ is an operator of type D on $\mathcal{D}(\Lambda^2) \cap X_+$, and this has the following important consequence:

If a positive solution f of Eq. (1.3) is sufficiently regular, i.e., if $f(t) \in \mathcal{D}(\Lambda^2)$, $t \geq 0$, a.e., and if $Q^{\pm}(\cdot, f(\cdot))$, $\Lambda Q^{\pm}(\cdot, f(\cdot)) \in L^1(\mathbb{R}_+; X_+)$, then by (2.10), applied with equality sign,

$$\|\Lambda f(t)\| + \int_0^t \Delta(s, f(s))ds = \|\Lambda f_0\| \quad (t \geq 0). \tag{3.5}$$

This shows that $\|\Lambda f(t)\|$ is decreasing in time and satisfies (1.4), as mentioned in Section 1. In particular, if $\Delta(t,g) = 0$ for all $g \in \mathcal{D}(\Lambda^2) \cap X_+$, $t \geq 0$ a.e., then $\|\Lambda f(t)\|$ is conserved for all $t \geq 0$.

Formula (3.5) is essential in our analysis, as a generalization of a priori estimates introduced in [1] and [5]-[8].

Next, observe that inequality (3.4) is of the form

$$-\Delta(t, g; \Gamma, Q) \leq \rho_\Gamma(\|\Lambda_1 g\|)\, \|\Gamma g\| \quad (g \in \mathcal{M}_1,\ t \geq 0\ a.e.), \tag{3.6}$$

where $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is some positive linear operator, and $\mathcal{M}_1 \subset \mathcal{D}(\Gamma) \cap \mathcal{D}(\Lambda^2) \cap X_+$ is such that $Q^{\pm}(t, \mathcal{M}_1) \subset \mathcal{D}(\Gamma)$, $t \geq 0$ a.e., while $\rho_\Gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is some non-decreasing function.

Formula (3.6) can be regarded as an abstract correspondent to the Povzner inequality [1], [3], and plays a key role in extending the argument of [1] to our analysis.

Here, it should be emphasized that the above setting does not exclude the case $\Lambda_1 = \Lambda$ when, obviously, some of the above conditions become redundant.

Under hypotheses (A_0)-(A_3), we can now state our main result:

Theorem 3.1. *Let either of the following two sets of conditions be fulfilled:*

a) $Q^+(t, \mathcal{D}_+^\infty) \subset \mathcal{D}_+^\infty$, $t \geq 0$ *a.e.*, $\Lambda^k Q^+(\cdot, \mathcal{D}_+^\infty) \subset L^1_{loc}(\mathbb{R}_+; X_+)$, $k = 1, 2, ...$ *In problem (1.1),* $f_0 \in \mathcal{D}(\Lambda^2) \cap X_+$.

b) The operators $Q^{\pm}$ *do not depend explicitly on* t. *In problem (1.1),* $f_0 \in \mathcal{D}(\Lambda^3) \cap X_+$.

Then there exists a unique positive strong solution of the i.v.p. (1.1) such that $f(t) \in \mathcal{D}(\Lambda^2)$ *for any* $t \geq 0$, *and* $\|\Lambda^2 f(\cdot)\|$ *is locally bounded on* $\mathbb{R}_+$.

Moreover, $f, \Lambda f \in C(\mathbb{R}_+; X_+)$. *Furthermore,* f *satisfies Eq. (3.5) and*

$$\|\Lambda^2 f(t)\| \leq \exp(\rho(\|\Lambda_1 f_0\|)t)\ \|\Lambda^2 f_0\| \quad (t \geq 0). \tag{3.7}$$

The proof of the above result will be given in the next section. Here we only remark that Theorem 3.1a) is also applicable to the autonomous case, but, clearly, its conditions are different from those of Theorem 3.1b).

The following proposition yields additional useful estimates for the solutions of Eq. (1.1). For simplicity, we remain in the conditions of Theorem3.1a). However, similar results are valid when Theorem3.1b) holds, as can be seen by inspecting the proof of the proposition, given in Section 4.

Assume that $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is a closed, positive linear operator. Let f be a solution of Problem (1.1), provided by Theorem 3.1a).

Proposition 3.1. *a) Suppose that Γ is of type D on $\mathcal{D}_+^\infty$. Then $f(t) \in \mathcal{D}(\Gamma)$, $t \geq 0$, and*

$$\|\Gamma f(t)\| \leq \|\Gamma f_0\| \quad (t \geq 0). \tag{3.8}$$

b) Suppose that Γ and ρ_Γ are as in (3.6), with $\mathcal{M}_1 \supseteq \mathcal{D}_+^\infty$. Then $f(t) \in \mathcal{D}(\Gamma)$, $t \geq 0$, and

$$\|\Gamma f(t)\| \leq \exp(\rho_\Gamma(\|\Lambda_1 f_0\|)t) \ \|\Gamma f_0\| \quad (t \geq 0). \tag{3.9}$$

It should be pointed out here, that in applications, the choice of Λ and Λ_1 may be not unique. In some cases, the role of Λ_1 and Γ may be played by suitable powers of Λ, while, in other examples, $\Lambda = \Lambda_1 = \Gamma$ (see Section 5).

Theorem 3.1 has an immediate noticeable consequence, in the following situation: Consider Eq. (1.5) and let $\{U^t\}_{t\in\mathbb{R}}$ be the C_0 group of positive linear isometries on X, generated by A.

We recall that any strong solution of Eq. (1.5) satisfies

$$f(t) = U^t f_0 + \int_0^t U^{t-s} Q(s, f(s))ds \quad (t \geq 0), \tag{3.10}$$

but the converse is not generally true.

We say that $f \in C(\mathbb{R}_+; X_+)$ is a mild solution of Eq. (1.5) if it satisfies Eq. (3.10). If f is a solution of (3.10), then setting $F(t) := U^{-t} f(t)$ in (3.10), we get

$$F(t) = f_0 + \int_0^t Q_U(s, F(s))ds \quad (t \geq 0),$$

hence, by differentiation,

$$\frac{d}{dt}F = Q_U(t, F) = Q_U^+(t, F) - Q_U^-(t, F). \quad F(0) = f_0, \quad (t \geq 0 \quad a.e.) \tag{3.11}$$

where $Q_U(t, \cdot) := U^{-t}Q(t, U^t\cdot)$ and $Q_U^\pm(t, \cdot) := U^{-t}Q^\pm(t, U^t\cdot)$.

Suppose that $U^t\mathcal{D}(\Lambda) = \mathcal{D}(\Lambda)$ and $U^t\Lambda = \Lambda U^t$ on $\mathcal{D}(\Lambda)$ for every $t > 0$. Similarly, assume that $U^t\mathcal{D}(\Lambda_1) = \mathcal{D}(\Lambda_1)$ and $U^t\Lambda_1 = \Lambda_1 U^t$ on $\mathcal{D}(\Lambda_1)$ for all $t > 0$.

Now $Q_U^\pm$ and Q_U are well defined as maps from $\mathbb{R}_+ \times \mathcal{D}(\Lambda)$ to X, the last equation is of the form (1.1), and we can state the following consequence of Theorem 3.1a):

Corollary 3.1. *Let $Q^+(t, \mathcal{D}_+^\infty) \subset \mathcal{D}_+^\infty$, $t \geq 0$ a.e., $\Lambda^k Q^+(\cdot, U^\cdot g) \in L^1_{loc}(\mathbb{R}_+; X_+)$ for all $g \in \mathcal{D}_+^\infty$, $k = 1, 2, \ldots..$ Suppose that $f_0 \in \mathcal{D}(\Lambda^2) \cap X_+$ in Eq. (1.5). Then Problem (1.5) has a unique positive mild solution f such that $f(t) \in \mathcal{D}(\Lambda^2)$ for any $t \geq 0$ and $\|\Lambda^2 f(\cdot)\|$ is locally bounded on $\mathbb{R}_+$. Moreover, $f, \Lambda f \in C(\mathbb{R}_+; X_+)$. Furthermore, f satisfies Eq. (3.5) and inequality (3.7).*

Proof. Note first that $U^t\mathcal{D}(\Lambda^k) = \mathcal{D}(\Lambda^k)$ and $U^t\Lambda^k g = \Lambda^k U^t g$ for all $g \in \mathcal{D}(\Lambda^k)$, $t \in \mathbb{R}$, $k = 1, 2, \ldots..$ In particular, $U^t\mathcal{D}_\infty^+ = \mathcal{D}_\infty^+$, for all $t \in \mathbb{R}$. These and the commutation properties of U^t with Λ_1 imply that the operators $Q_U^\pm$ satisfy the general conditions (domain conditions, measurability, isotonicity, o-closedness, and p-saturation) imposed on $Q^\pm$ by Theorem 3.1a). Further, it is straightforward to check that the conditions of (A_0)-(A_3) for the triplet (Λ, Q, a) are also satisfied by (Λ, Q_U, a). Indeed, if $g \in \mathcal{D}(\Lambda^2) \cap X_+$, then $\Delta(t, g; \Lambda, Q_U) = \Delta(t, U^t g)$ and $\Delta(t, g; \Lambda_1, Q_U) = \Delta(t, U^t g; \Lambda_1, Q)$, $t \geq 0$ a.e. Moreover,

$$a(\|\Lambda g\|)\Lambda g - Q_U^-(t, g) = U^{-t}\left[a(\|\Lambda U^t g\|)\Lambda U^t g - Q^-(t, U^t g)\right], \quad g \in \mathcal{D}(\Lambda) \cap X_+$$

for a.e. $t \geq 0$. Finally, by inequality (3.4), we get

$$-\Delta(t, g; \Lambda^2, Q_U) = -\Delta(t, U^t g; \Lambda^2, Q) \leq \rho\left(\|\Lambda_1 U^t g\|\right) \|\Lambda^2 U^t g\| = \rho\left(\|\Lambda_1 g\|\right) \|\Lambda^2 g\|$$

for all $g \in \mathcal{D}(\Lambda^3) \cap X_+$; $t \geq 0$ a.e. □

A correspondent to Proposition 3.1, applicable to Corollary 3.1, can be readily obtained. The modifications in the reformulation of the proposition are obvious and include additional hypotheses for the commutation of U^t with Γ and Γ_n.

We end this section with a few considerations, providing an insight into the argument of Theorem 3.1, and explaining the role of assumptions (A_0)-(A_3).

Being interested in the existence of strong positive solutions of Eq. (1.1), we would like to solve the equation by iteration. Unfortunately, this cannot be directly done, because of the form of Q. However, we can start from an equivalent problem, inspired from [1], as follows:

Formula (3.5) implies that any strong positive solution of Eq. (1.1) (in the regularity class for which (3.5) is valid) is also a solution to the problem

$$\frac{d}{dt}f + a(\|\Lambda f_0\|)\Lambda f = B(t,f,f), \quad f(0) = f_0 \in X_+ \quad (t \geq 0). \tag{3.12}$$

Here a is as in (A_2), and B is formally defined by

$$B(t,g,h) := Q(t,g(t)) + a\left(\|\Lambda g(t)\| + \int_0^t \Delta(s,h(s))ds\right)\Lambda g(t), \quad (t \geq 0 \quad a.e.) \tag{3.13}$$

for all $g(t) \in \mathcal{D}(\Lambda) \cap X_+$ and $h(t) \in \mathcal{D}(\Lambda^2) \cap X_+$ with $\Lambda Q^{\pm}(\cdot, h(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$.

Conversely, any positive strong solution of Problem (3.12) is a solution of Eq. (1.1), provided that it satisfies (3.5).

Further, by (A_0) and Lemma 2.1b), the operator $L = -a(\|\Lambda f_0\|)\Lambda$ is the infinitesimal generator of a C_0 positive semigroup $\{V^t\}_{t\geq 0}$, and

$$0 \leq V^t h \leq \exp(-a(\|\Lambda f_0\|)\lambda_0 t)h \leq h \quad (h \in X_+). \tag{3.14}$$

Then, any solution of Eq. (3.12) is also a solution of the mild problem

$$f(t) = V^t f_0 + \int_0^t V^{t-s}B(s,f,f)ds\,, \tag{3.15}$$

the integral being in the sense of Bochner.

The last equation presents some advantages for monotone iteration. Indeed, as will appear later, $g \mapsto B(t,g,h)$ and $h \mapsto B(t,g,h)$ define positive and isotone mappings. Then, as $\{V^t\}_{t\geq 0}$ is positive, the iteration

$$f_1(t) = 0, \quad f_2(t) = V^t f_0,$$

$$f_n(t) = V^t f_0 + \int_0^t V^{t-s}B(s,f_{n-1},f_{n-2})ds \quad (n = 3,4,...) \tag{3.16}$$

is positive and increasing.

Recall now that X is monotone complete. Then to show that the sequence $\{f_n(t)\}$ is convergent, it is sufficient to prove that it is norm-bounded. To this end, one can hopefully use the dissipation property (3.2). The limit $f^*(t)$ of $\{f_n(t)\}$ is expected to satisfy (3.15), i.e., to be a mild solution of Eq. (3.12). Moreover, under suitable regularity conditions on f_0, one can actually find that $f^*(t)$ is also a strong solution of Eq. (3.12). Finally, to show that $f^*(t)$ is a strong solution of Eq. (1.1), one needs only prove that it satisfies Eq. (3.5). To this end, one can use (3.4).

4. Proofs of Theorem 3.1 and Proposition 3.1

The main difficulty behind the argument of Theorem 3.1 comes from the nonlinear and singular nature of $Q^{\pm}$. Thus, before studying the convergence of the iteration (3.16), we need a rather careful domain compatibility and regularity analysis, to ensure the consistency of the iteration (3.16). To this end, we first investigate the properties of $Q^{\pm}$ and of the operator B formally defined in (3.13).

We start with some simple inequalities.

Let $g \in \mathcal{D}(\Lambda^2) \cap X_+$. Then (3.1) gives

$$\|g\| \leq \lambda_0^{-1} \|\Lambda g\| \leq \lambda_0^{-2} \left\|\Lambda^2 g\right\|. \tag{4.1}$$

In addition, applying (3.1) to $Q^{\pm}(t,g)$, using (3.2), (3.3), and again (3.1), we get

$$\left\|Q^{\pm}(t,g)\right\| \leq \lambda_0^{-1} \left\|\Lambda Q^{\pm}(t,g)\right\| \leq \lambda_0^{-1} \left\|\Lambda Q^{-}(t,g)\right\|$$
$$\leq a(\|\Lambda g\|)\lambda_0^{-1} \left\|\Lambda^2 g\right\| \leq a(\lambda_0^{-1} \left\|\Lambda^2 g\right\|)\lambda_0^{-1} \left\|\Lambda^2 g\right\| \quad (t \geq 0 \quad a.e.). \tag{4.2}$$

Notice the following obvious consequences of (4.1) and (4.2).

Remark 4.1. *$Q^{\pm}(t,0) = 0$ and $\Delta(t,0) = 0$ a.e. on $\mathbb{R}_+$.*

Let $\Lambda^0 := I$.

Remark 4.2. *If $g : \mathbb{R}_+ \mapsto X_+$ is measurable, with $g(t) \in \mathcal{D}(\Lambda^2)$, $t \geq 0$, a.e., and $\left\|\Lambda^2 g\right\| \in L^\infty_{loc}(\mathbb{R}_+)$, then g, $\Lambda^{k+1} g$, and $\Lambda^k Q^{\pm}(\cdot, g(\cdot))$ are in $L^1_{loc}(\mathbb{R}_+; X_+)$, $k = 0, 1$.*

The next result makes precise the monotonicity properties of B.

Lemma 4.1. *Let g_i, h_i, $i = 1, 2$, satisfy the conditions of Remark 4.2. Suppose that $g_1(t) \leq g_2(t)$ and $h_1(t) \leq h_2(t)$ a.e. on $\mathbb{R}_+$. Then $B(\cdot, g_i, h_j) \in L^1_{loc}(\mathbb{R}_+; X_+)$, $i, j = 1, 2$. In addition, for a.e. $t \geq 0$,*

$$0 \leq B(t, g_1, h_1) \leq B(t, g_2, h_2). \tag{4.3}$$

Proof. The first assertion in Lemma 4.1 is immediate from Remark 4.2, and from assumptions (A_1) and (A_2).

To prove (4.3), define

$$y_i(t) := \int_0^t \Delta(s, h_i(s))ds \quad (i = 1, 2), \tag{4.4}$$

and observe that $0 \leq y_1(t) \leq y_2(t)$, because of the isotonicity of $\Delta(t, \cdot)$ (cf. (A_1)).

Next, as a is non-decreasing (cf. (A_2)), clearly $F(x,y) := a(x+y) - a(x) \geq 0$ for all $x, y \geq 0$. Besides, for each $x \geq 0$, the function $\mathbb{R}_+ \ni y \mapsto F(x,y) \in \mathbb{R}_+$ is non-decreasing. Since a is non-decreasing and convex (cf. (A_2)), it follows that for each $y \geq 0$, the function $\mathbb{R}_+ \ni x \mapsto F(x,y) \in \mathbb{R}_+$ is also non-decreasing: indeed, the derivative a' of a is a.e. well defined, positive and non-decreasing, hence,

$$F(x^*, y) - F(x, y) = \int_0^y \left[a'(x^* + \xi) - a'(x + \xi)\right] d\xi \geq 0 \tag{4.5}$$

for all $0 \leq x \leq x^*$ and $y \geq 0$.

Further observe that $0 \leq B(t, g_1, 0) \leq B(t, g_2, 0)$, by (A_2) and the isotonicity of $Q^+(t, \cdot)$. Then, the definition of F and the obvious inequality $\Lambda g_1(t) \leq \Lambda g_2(t)$ give

$$0 \leq B(t, g_1, h_1) = B(t, g_1, 0) + F\left(\|\Lambda g_1(t)\|, y_1(t)\right) \Lambda g_1(t)$$
$$\leq B(t, g_2, 0) + F\left(\|\Lambda g_1(t)\|, y_1(t)\right) \Lambda g_2(t). \tag{4.6}$$

But the monotonicity properties of F imply the inequalities

$$0 \leq F\left(\|\Lambda g_1(t)\|, y_1(t)\right) \leq F\left(\|\Lambda g_2(t)\|, y_1(t)\right) \leq F\left(\|\Lambda g_2(t)\|, y_2(t)\right),$$

which, applied in (4.6), lead immediately to (4.3). □

The next two lemmas give a precise meaning to $f_n(t)$ formally defined by iteration (3.16), and refer to some useful properties of the iteration.

Lemma 4.2. *Let n=1,2,....*

a) Under the conditions of Theorem 3.1a), let $f_0 \in \mathcal{D}_+^\infty$. *Then* $f_n(t)$, $Q^\pm(t, f_n(t)) \in \mathcal{D}_+^\infty$ *a.e. on* $\mathbb{R}_+$. *Moreover,* $\Lambda^k Q^\pm(\cdot, f_n(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$, $k = 0, 1, \ldots$.

b) Assume the conditions of Theorem 3.1b). Then $f_n(t) \in \mathcal{D}(\Lambda^3) \cap X_+$ *and* $Q^\pm(f_n(t)) \in \mathcal{D}(\Lambda^2) \cap X_+$; $t \geq 0$. *Moreover,* $\Lambda^k Q^\pm(f_n) \in L^1_{loc}(\mathbb{R}_+; X_+)$, $k = 0, 1, 2$.

Furthermore, in both cases a) and b), $\Lambda^k f_n \in C(\mathbb{R}_+; X_+)$, $k = 0, 1, 2$, *and* f_n *is a.e. differentiable on* $\mathbb{R}_+$.

Proof. a) It is sufficient to show that for each $T > 0$ and $n = 1, 2, \ldots$, there is $g_{n,T} \in \mathcal{D}_+^\infty$ such that

$$0 \leq f_n(t) \leq g_{n,T} \quad (0 \leq t \leq T \quad a.e.). \tag{4.7}$$

Indeed, as $\mathcal{D}_+^\infty$ is p-saturated, applying (3.3) we find that $Q^-(t, g_{n,T}) \in \mathcal{D}_+^\infty$ a.e. on $\mathbb{R}_+$, and $\Lambda^k Q^-(\cdot, g_{n,T}) \in L^1_{loc}(\mathbb{R}_+; X_+)$ for all $k = 0, 1, 2, \ldots$. The same properties are verified by $Q^+(t, g_{n,T})$ and $\Lambda^k Q^+(\cdot, g_{n,T})$, respectively, as it follows from the assumptions of Theorem 3.1a) and (4.2). Then (4.7) implies immediately the statement a) of the above lemma, because $\mathcal{D}_+^\infty$ is p-saturated and the operators $\Lambda^k Q^\pm(t, \cdot)$ are positive and isotone for a.e. $t \geq 0$; $k = 0, 1, \ldots$.

It remains therefore to prove (4.7). To this end, we proceed by induction.

First set $g_{1,T} := 0$ and $g_{2,T} := f_0$. Then (4.7) is trivially verified for $n = 1, 2$.

Next, let $q \geq 3$ and $T > 0$ be fixed (but arbitrary). Suppose that for each $n = 1, 2, .., q-1$, there is some $g_{n,T} \in \mathcal{D}_+^\infty$ that satisfies (4.7). By the above considerations and the properties of Δ and a, clearly $B(t, g_{n,T}, g_{n-1,T}) \in \mathcal{D}_+^\infty$, $0 \leq t \leq T$ a.e., and $\Lambda^k B(\cdot, g_{n,T}, g_{n-1,T}) \in L^1(0, T; X_+)$, $k = 1, 2, \ldots$. Then, as Λ^k is closed, we can take advantage of (2.1) to write

$$\Lambda^k \int_0^t B(s, g_{n-1,T}, g_{n-2,T}) ds = \int_0^t \Lambda^k B(s, g_{n-1,T}, g_{n-2,T}) ds \quad (0 \leq t \leq T). \tag{4.8}$$

for all $k = 1, 2, \ldots$ and $n = 1, 2, \ldots, q-1$.

Now observe that $f_{q-1}(t) \leq g_{q-1,T}$ and $f_{q-2}(t) \leq g_{q-2,T}$ satisfy the conditions of Lemma 4.1 for $g_1 \leq g_2$ and $h_1 \leq h_2$, respectively. Then applying conveniently (3.14) and (4.3) in (3.16), and invoking (4.8), we get

$$0 \leq f_q(t) \leq f_0 + \int_0^T B(s, g_{q-1,T}, g_{q-2,T}) ds := g_{q,T} \in \mathcal{D}_+^\infty \quad (0 \leq t \leq T). \tag{4.9}$$

This concludes the induction argument and the proof of a) (as $T > 0$ is arbitrary).

b) It is sufficient to show that property (4.7) is verified by $g_{n,T} \in \mathcal{D}(\Lambda^3) \cap X_+$. Indeed, if $g_{n,T} \in \mathcal{D}(\Lambda^3) \cap X_+$, then, evidently, $Q^\pm(g_{n,T})$ is time-independent. Moreover, the hypotheses on $Q^\pm$ give $Q^\pm(g_{n,T}) \in \mathcal{D}(\Lambda^2) \cap X_+$. Consequently, trivially (in the autonomous case), $\Lambda^k Q^\pm(g_{n,T}) \in L^1(0, T; X_+)$, $k = 0, 1, 2$.

As before, if $g_{1,T} = 0$ and $g_{2,T} = f_0$, then (4.7) is trivially verified for $n = 1, 2$.

Let $q \geq 3$ and $T > 0$ be fixed. Suppose that for each $n = 1, 2, \ldots, q-1$, there is some $g_{n,T} \in \mathcal{D}(\Lambda^3) \cap X_+$ that satisfies (4.7). Then $B(t, g_{q-1,T}, g_{q-2,T}) \in \mathcal{D}(\Lambda^2) \cap$

X_+, $0 \le t \le T$, because $Q^{\pm}(g_{n,T}) \in \mathcal{D}(\Lambda^2) \cap X_+$. But the semigroup V^t is of class C_0, with the infinitesimal generator $L = -a(\|\Lambda f_0\|)\Lambda$, hence $\int_0^t V^s h ds \in \mathcal{D}(\Lambda)$ for all $h \in X$, $t \ge 0$. Consequently, for any $0 \le t \le T$,

$$\int_0^t V^{t-s} B(T, g_{q-1,T}, g_{q-2,T}) ds = \int_0^t V^s B(T, g_{q-1,T}, g_{q-2,T}) ds \in \mathcal{D}(\Lambda^3) \cap X_+. \tag{4.10}$$

As a is positive and non-decreasing,

$$B(t, g_{q-1,T}, g_{q-2,T}) \le B(T, g_{q-1,T}, g_{q-2,T}) \quad (0 \le t \le T). \tag{4.11}$$

Further, making use of Lemma 4.1, as in the proof of a), applying again (3.14) and (4.3) in (3.16), and combining with (4.11), we obtain

$$0 \le f_q(t) \le f_0 + \int_0^t V^{t-s} B(T, g_{q-1,T}, g_{q-2,T}) ds \quad (0 \le t \le T). \tag{4.12}$$

Now invoking (4.10), we are led to

$$f_q(t) \le f_0 + \int_0^T V^s B(T, g_{q-1,T}, g_{q-2,T}) ds := g_{q,T} \in \mathcal{D}(\Lambda^3) \cap X_+ \quad (0 \le t \le T), \tag{4.13}$$

concluding the proof of b) (since T is arbitrary).

To end the proof of the lemma, note in case a) that $\Lambda^k B(\cdot, g_{n,T}, g_{n-1,T}) \in L^1_{loc}(\mathbb{R}_+; X_+)$, hence by (4.3), we get $\Lambda^k B(\cdot, f_{n-1}, f_{n-2}) \in L^1_{loc}(\mathbb{R}_+; X_+)$; $k = 0, 1, 2, \ldots$. In case b), if $k = 0, 1, 2$, then $B(T, g_{n-1,T}, g_{n-2,T}) \in \mathcal{D}(\Lambda^2) \cap X_+$, so that (4.3) and (4.11) give $\Lambda^k B(\cdot, f_{n-1}, f_{n-2}) \in L^1_{loc}(\mathbb{R}_+; X_+)$, $k = 0, 1, 2$. As Λ^k is closed, one can then make use of (2.1), and find that Λ^k commutes with the Bochner integral, when applied to (3.16), $k = 1, 2$. This implies $\Lambda^k f_n \in C(0, T; X_+)$, $k = 0, 1, 2$.

Finally, to show that the r.h.s. of (3.16) is a.e. differentiable on $\mathbb{R}_+$, we simply take advantage that $L = -a(\|\Lambda f_0\|)\Lambda$ is the infinitesimal generator of V^t. □

The sequence $\{f_n(t)\}$ was so far expected to approximate the solutions of Eq. (3.15). Under the conditions of Lemma 4.2, as $f_n(t)$ is differentiable, we could consider the sequence $\{f_n(t)\}$ to approximate the strong solutions of Eq. (3.12) (and of Eq. (1.1)). To this end, we start by differentiating (3.16), and obtain

$$\frac{d}{dt} f_n(t) = B(t, f_{n-1}, f_{n-2}) - a(\|\Lambda f_0\|)\Lambda f_n(t) \quad (t > 0 \quad a.e., \quad n \ge 3). \tag{4.14}$$

Integrating again Eq. (4.14), we obtain an equivalent formula, valid for $n \ge 3$,

$$f_n(t) = f_0 + \int_0^t Q(s, f_{n-1}(s)) ds$$
$$+ \int_0^t \left[a\left(\|\Lambda f_{n-1}(s)\| + \int_0^s \Delta(\tau, f_{n-2}(\tau)) d\tau \right) \Lambda f_{n-1}(s) - a(\|\Lambda f_0\|)\Lambda f_n(s) \right] ds. \tag{4.15}$$

Lemma 4.3. *If f_n is as in Lemma 4.2, then for any $t \ge 0$, the sequence $\{f_n(t)\}$ is increasing. Moreover, if $n \ge 2$, then*

$$f_n(t) \le f_0 + \int_0^t Q(s, f_{n-1}(s)) ds \tag{4.16}$$

and

$$\|\Lambda f_n(t)\| + \int_0^t \Delta(s, f_{n-1}(s))ds \le \|\Lambda f_0\| . \tag{4.17}$$

Proof. Evidently, $0 = f_1(t) \le f_2(t) \le f_3(t)$ a.e., and a straightforward induction, applying (4.3), shows that $\{f_n(t)\}$ is a.e. increasing.

For the rest of the proof, first note that (4.16) implies (4.17): Indeed, (4.16) is of the form (2.9) (with Γ, g_0, g, h replaced by $\Lambda, f_0, f_n, f_{n-1}$). Then Lemma 2.2 applies, because Λ is of type D on $\mathcal{D}(\Lambda^2)\cap X_+$, and $\Lambda Q^{\pm}(\cdot, f_{n-1}(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$ by Remark 4.2.

Therefore, it remains only to prove (4.16). We proceed again by induction.

Since $0 = f_1 \le f_2(t) \le f_0$, and $\Delta(t, 0) = 0$ a.e. (cf. Remark 4.1), it appears that (4.16) is trivially verified for $n = 2$.

Let $q \ge 3$ and suppose inequality (4.16) to be valid for $n = 2, 3, ..., q-1$.

If $n = q$ in (4.15), applying the positivity of a and the obvious property $0 \le \Lambda f_{q-1}(t) \le \Lambda f_q(t)$, we get

$$f_q(t) \le f_0 + \int_0^t Q(s, f_{q-1}(s))ds$$

$$+ \int_0^t \left[a\left(\|\Lambda f_{q-1}(s)\| + \int_0^s \Delta(\tau, f_{q-2}(\tau))d\tau \right) - a\left(\|\Lambda f_0\|\right) \right] \Lambda f_q(s)ds. \tag{4.18}$$

According to the induction hypothesis, (4.16) holds true for $n = q-1$. Hence (4.17) is also valid for $n = q-1$, as concluded before. Then $a(\|\Lambda f_{q-1}(s)\| + \int_0^s \Delta(\tau, f_{q-2}(\tau))d\tau) \le a\left(\|\Lambda f_0\|\right)$, because a is non-decreasing. As $\Lambda f_q(s)$ is positive, clearly the integral term containing $\Lambda f_q(s)$, in the r.h.s. of (4.18) is negative. Then (4.16) becomes true for $n = q$. This concludes the proof of the lemma. □

Let $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ be a closed, positive linear operator. If Γ is of type D, or satisfies (3.6), and f_n is as in Lemma 4.2, one can characterize $\|\Gamma f_n(t)\|$ as follows:

Lemma 4.4. *a) Under the conditions of Theorem 3.1a) (Theorem 3.1b)), if Γ is of type D on $\mathcal{D}^{\infty}_{+}$, (on $\mathcal{D}(\Lambda^2) \cap X_+$) then for any $t \ge 0$,*

$$\|\Gamma f_n(t)\| \le \|\Gamma f_0\| \quad (n = 1, 2, ...). \tag{4.19}$$

b) Under the conditions of Theorem 3.1a) (Theorem 3.1b)), suppose that Γ satisfies (3.6) with $\mathcal{M}_1 \supseteq \mathcal{D}^{\infty}_{+}$ (with $\mathcal{M}_1 \supseteq \mathcal{D}(\Lambda^3) \cap X_+$). Then for any $t \ge 0$,

$$\|\Gamma f_n(t)\| \le \exp(\rho_\Gamma(\|\Lambda_1 f_0\|)t) \|\Gamma f_0\| \quad (n = 1, 2, ...). \tag{4.20}$$

with ρ_Γ as in (3.6).

Proof. Lemma 4.2 implies that $Q^{\pm}(t, f_n(t)) \in \mathcal{D}(\Gamma)$, for a.e. $t \ge 0$. Moreover, $\Gamma Q^{\pm}(\cdot, f_n(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$. Indeed, let $T > 0$ and $g_{n,T} \ge f_n(t)$ be as in Lemma 4.2. If Γ is of type D on $\mathcal{D}^{\infty}_{+}$ (on $\mathcal{D}(\Lambda^2) \cap X_+$), then by (2.8) and (3.3), we obtain $\|\Gamma Q^{\pm}(t, f_n(t))\| \le \|\Gamma Q^{\pm}(t, g_{n,T})\| \le \|\Gamma Q^{-}(t, g_{n,T})\| \le a(\|g_{n,T}\|) \|\Gamma \Lambda g_{n,T}\|$ for a.e. $0 \le t \le T$. On the other hand, if Γ satisfies (3.6), then (3.3) implies $\|\Gamma Q^{+}(t, f_n(t))\| \le \|\Gamma Q^{-}(t, f_n(t))\| + \rho_\Gamma(\|\Lambda_1 g_{n,T}\|) \|\Gamma g_{n,T}\| \le a(\|g_{n,T}\|) \|\Gamma \Lambda g_{n,T}\| + \rho_\Gamma(\|\Lambda_1 g_{n,T}\|) \|\Gamma g_{n,T}\|$, $0 \le t \le T$, a.e.

But (4.16) is of the form (2.9), and the above considerations show that Lemma 2.2 applies (with Γ instead of Λ). Hence,

$$\|\Gamma f_n(t)\| + \int_0^t \Delta(s, f_{n-1}(s); \Gamma, Q)ds \le \|\Gamma f_0\| \quad (t \ge 0, \quad n \ge 2). \tag{4.21}$$

Now the proof of a) can be immediately concluded: if $n = 1$, then formula (4.19) is trivially satisfied; if $n \geq 2$, then (4.19) is directly implied by (4.21).

To prove statement b), first apply inequality (3.6) in (4.21). It follows that

$$\|\Gamma f_n(t)\| \leq \|\Gamma f_0\| + \int_0^t \rho_\Gamma(\|\Lambda_1 f_{n-1}(s)\|) \, \|\Gamma f_{n-1}(s)\| \, ds \quad (t \geq 0, \quad n \geq 2). \quad (4.22)$$

But Λ_1 satisfies the conditions of a) in the present lemma, hence $\|\Lambda_1 f_n(t)\| \leq \|\Lambda_1 f_0\|$, $t \geq 0$, $n = 1, 2 \ldots$. Introducing the last inequality in (4.22), we obtain

$$\|\Gamma f_n(t)\| \leq \|\Gamma f_0\| + \rho_\Gamma(\|\Lambda_1 f_0\|) \int_0^t \|\Gamma f_{n-1}(s)\| \, ds \quad (t \geq 0, \quad n \geq 2). \qquad (4.23)$$

Finally, since (4.20) is obviously satisfied for $n = 1, 2$, a straightforward Gronwall type induction in (4.23) concludes the proof of b). □

Remark 4.3. *Since Λ^2 satisfies the conditions for Γ in Lemma 4.4b), it follows that*

$$\left\|\Lambda^2 f_n(t)\right\| \leq \exp(\rho(\|\Lambda_1 f_0\|)t) \left\|\Lambda^2 f_0\right\| \quad (t \geq 0, \quad n = 1, 2, \ldots.) \qquad (4.24)$$

with ρ as in (3.4).

Proof of Theorem 3.1 . We start by observing that if $f_0 = 0$ in Problem (1.1), then $f(t) \equiv 0$ is a solution to Eq. (1.1), as an immediate consequence of Remark 4.1. It is the unique strong solution in $\mathcal{D}(\Lambda^2) \cap X_+$, as it follows from formula (3.5).

Moreover, if $0 \neq f_0 \in \mathcal{D}(\Lambda^2) \cap X_+$, but $a(\|\Lambda f_0\|) = 0$, then $Q^\pm(t, f_0) = 0$, for a.e. $t \geq 0$, by (4.2), hence $f(t) \equiv f_0$ is a solution to Problem (1.1). Its uniqueness in the class of solutions in $\mathcal{D}(\Lambda^2) \cap X_+$ is immediate because any other solution $f^*(t) \in \mathcal{D}(\Lambda^2) \cap X_+$ must be a.e. constant: indeed, applying formula (3.5), and invoking the positivity and monotonicity of a, we obtain $0 \leq a(\|\Lambda f^*(t)\|) \leq a(\|\Lambda f_0\|) = 0$. This leads (again by (4.2)) to $Q^\pm(t, f(t)) = 0$ a.e.

Therefore, we can assume below that $f_0 \neq 0$ and $a(\|\Lambda f_0\|) \neq 0$.

First we prove the existence part of the theorem.

Existence in case a). Step 1: Let $f_0 \in \mathcal{D}^\infty_+$. Then Lemmas 4.2, 4.3 and Remark 4.3 apply, hence f_n (defined by (3.16)) satisfies (4.24). Then formula (3.1), the monotone completeness of X, and the fact that Λ^k is closed imply that there is $f(t) \in \mathcal{D}(\Lambda^k)$ such that $\Lambda^k f_n(t) \nearrow \Lambda^k f(t)$ as $n \to \infty$, $t \geq 0$, $k = 0, 1, 2$. Consequently, $f(t)$ satisfies (3.7). Moreover, Remark 4.2 implies that $\Lambda^k f$, $k = 0, 1, 2$, $Q^\pm(\cdot, f(\cdot))$, and $\Lambda Q^\pm(\cdot, f(\cdot))$ are in $L^1_{loc}(\mathbb{R}_+; X_+)$. Then, one can apply the Lebesgue's dominated convergence theorem in (4.15) and (4.17). It follows that

$$f(t) = f_0 + \int_0^t Q(s, f(s))ds$$

$$+ \int_0^t \left[a\left(\|\Lambda f(s)\| + \int_0^s \Delta(\tau, f(\tau))d\tau\right) - a(\|\Lambda f_0\|)\right] \Lambda f(s)ds \quad (t \geq 0) \qquad (4.25)$$

(i.e., f is a strong solution of Eq.(3.12)) and

$$0 \leq \psi(t) := \|\Lambda f_0\| - \|\Lambda f(t)\| - \int_0^t \Delta(s, f(s))ds \quad (t \geq 0). \qquad (4.26)$$

Obviously, (4.25) implies that $f \in C(\mathbb{R}_+; X_+)$. By (2.1), the operator Λ commutes with the integral, when applied to (4.25). Consequently, $\Lambda f \in C(\mathbb{R}_+; X_+)$.

As it appears from formula (4.25), to prove that f is a strong solution of (1.1), it is sufficient to show that $\psi \equiv 0$ (which means exactly (3.5)).

To this end we rewrite Eq. (4.25) conveniently, and apply Λ to the resulting equation. We get

$$\Lambda f_0 + \int_0^t \Lambda Q^+(s, f(s))ds = \Lambda f(t) + \int_0^t \Lambda Q^-(s, f(s))ds +$$

$$+ \int_0^t \left[a(\|\Lambda f_0\|) - a\left(\|\Lambda f(s)\| + \int_0^s \Delta(\tau, f(\tau))d\tau \right) \right] \Lambda^2 f(s)ds. \tag{4.27}$$

On the other hand, as a is non-decreasing, formula (4.26) implies

$$a(\|\Lambda f_0\|) \geq a\left(\|\Lambda f(t)\| + \int_0^t \Delta(s, f(s))d\tau \right), \tag{4.28}$$

thus all the integrands in (4.27) are positive. Then we can apply (1.2) and (2.2) as in the proof of Lemma 2.2. We obtain

$$\psi(t) = \int_0^t \left[a(\|\Lambda f_0\|) - a\left(\|\Lambda f(s)\| + \int_0^s \Delta(\tau, f(\tau))d\tau \right) \right] \left\|\Lambda^2 f(s)\right\| ds. \tag{4.29}$$

But the function a is the non-decreasing and convex (cf. (A_2)). Then it is locally Lipschitz, and, by (4.26) and (4.28), there is a number $0 < c = c(\|\Lambda f_0\|)$, depending only on $\|\Lambda f_0\|$, such that

$$0 \leq a(\|\Lambda f_0\|) - a\left(\|\Lambda f(t)\| + \int_0^t \Delta(\tau, f(\tau))d\tau \right) < c\psi(t). \tag{4.30}$$

As shown before, $f(t)$ satisfies (3.7). Then introducing (4.30) in (4.29), we find

$$0 \leq \psi(t) \leq c \int_0^t \psi(s) \left\|\Lambda^2 f(s)\right\| ds \leq c_T \int_0^t \psi(s)ds \quad (0 \leq t \leq T).$$

for each $T > 0$. Here, $c_T > 0$ is a number depending only on T and f_0.

Now the Gronwall inequality implies $\psi(t) = 0$, $0 \leq t \leq T$. As T is arbitrary, the existence part of the proof of Theorem 3.1a) is thus concluded in the case $f_0 \in \mathcal{D}_+^\infty$.

Step 2: Let f_0 be as Theorem 3.1a), i.e., $f_0 \in \mathcal{D}(\Lambda^2) \cap X_+$. By Lemma 2.1b), we can chose an increasing sequence $\{f_{0,i}\} \subset \mathcal{D}_+^\infty$, of initial conditions in Problem (1.1) such that $f_{0,i} \nearrow f_0$, as $i \to \infty$. Then, according to Step 1, we obtain a sequence of strong solutions $\{F_i\}$ of Eq. (1.1) with $F_i(0) = f_{0,i}$, satisfying the properties of the theorem. In particular,

$$\left\|\Lambda^2 F_i(t)\right\| \leq \exp\left[\rho(\|\Lambda_1 f_{0,i}\|)\right] \left\|\Lambda^2 f_{0,i}\right\| \quad (t \geq 0). \tag{4.31}$$

In addition,

$$F_i(t) = f_{0,i} + \int_0^t Q(s, F_i(s))ds, \tag{4.32}$$

$$\Lambda F_i(t) = \Lambda f_{0,i} + \int_0^t \Lambda Q(s, F_i(s))ds, \tag{4.33}$$

and

$$\|\Lambda F_i(t)\| + \int_0^t \Delta(s, F_i(s))ds = \|\Lambda f_{0,i}\|. \tag{4.34}$$

According to Step 1, each F_i is the limit of an increasing sequence $\{f_{n,i}(t)\}_{n=1}^\infty$ defined by (3.16) with $f_{n,i}(0) = f_{0,i}$. It can be easily seen from the positivity of V^t

and from Lemma 4.1 that if $f_{0,i} \leq f_{0,j}$ then $f_{n,i}(t) \leq f_{n,j}(t)$ for all n and $t \geq 0$. This implies that the sequence $\{F_i\}$ is increasing.

Moreover, since $\|\Lambda_1 f_{0,i}\| \leq \|\Lambda_1 f_0\|$, $\|\Lambda^2 f_{0,i}\| \leq \|\Lambda^2 f_0\|$, and since ρ is non-decreasing, it follows from inequality (4.31) that

$$\|\Lambda^2 F_i(t)\| \leq \exp(\rho(\|\Lambda_1 f_0\|)t) \|\Lambda^2 f_0\| \quad (t \geq 0). \tag{4.35}$$

Now a convergence argument, as in the beginning of Step 1, implies that there is an element $f \in L^1_{loc}(\mathbb{R}_+; X_+)$, with the properties stated in Remark 4.2, such that $F_i(t) \nearrow f(t)$ as $i \to \infty$, a.e. Consequently, we can apply, say, Lebesgue's convergence theorem in (4.32)-(4.34) to conclude the existence part of Theorem 3.1a).

Existence in case b). First notice that, in this case, Lemma 4.2 applies, corresponding to the fulfillment of the conditions of Theorem 3.1b). Then, the proof is as in Step 1 of case a).

Uniqueness. Remark that, applying the Lebesgue's convergence theorem to formula (3.16), we find that f is also a solution to Eq. (3.15). On the other hand, if F is another positive solution of Eq. (1.1) with the regularity stated in Theorem 3.1, then F satisfies Eq. (3.12), hence is also a solution to (3.15). But, because of the form of the iteration (3.16), it can be easily checked that $f \leq F$, for any other positive solution F of (3.15). Suppose now that such a solution $F \in C(\mathbb{R}_+; X_+)$, different from f, exists and verifies Eq. (3.5). Then

$$\|\Lambda f(t)\| + \int_0^t \Delta(s, f(s))ds = \|\Lambda f_0\| = \|\Lambda F(t)\| + \int_0^t \Delta(s, F(s))ds \tag{4.36}$$

However, f must differ from F on a subset of $\mathbb{R}_+$ with nonzero Lebesgue measure, where necessarily, $f(t) < F(t)$. Therefore

$$\|\Lambda f(t)\| + \int_0^t \Delta(s, f(s))ds < \|\Lambda F(t)\| + \int_0^t \Delta(s, F(s))ds.$$

on that subset, because Λ and $\|\cdot\|$ are strictly isotone mappings, and Δ is isotone. This contradicts (4.36), hence the uniqueness part of the proof is concluded. □

Proof of Proposition 3.1. a) Let f_0, $\{f_{0,i}\}$, $\{f_{n,i}(t)\}_{n=1}^{\infty}$, and $\{F_i(t)\}$ be as in Step 2 of the proof of Theorem 3.1a). Then for each i, the sequence $\{\Gamma f_{n,i}(t)\}_{n=1}^{\infty}$ is positive and increasing. Moreover, it is norm-bounded because

$$\|\Gamma f_{n,i}(t)\| \leq \|\Gamma f_0\| \quad (t \geq 0), \tag{4.37}$$

as a consequence of Lemma 4.4a) and of the property $\Gamma f_{0,i} \leq \Gamma f_0$.

As X is monotone complete, it follows that $\{\Gamma f_{n,i}(t)\}_{n=1}^{\infty}$ is convergent for all i.

Recall that Γ is closed, and $f_{n,i}(t) \nearrow F_i(t)$ as $n \to \infty$, for all i. Consequently, $F_i(t) \in \mathcal{D}(\Gamma)$ and $\Gamma f_{n,i}(t) \nearrow \Gamma F_i(t)$ as $n \to \infty$, $i = 1, 2, \ldots$. In addition, $\|\Gamma F_i\| \leq \|\Gamma f_0\|$, $t \geq 0, i = 1, 2, \ldots$. Then, reasoning as before, we conclude that $f(t) \in \mathcal{D}(\Gamma)$, $\Gamma F_i(t) \nearrow \Gamma f(t)$ as $i \to \infty$, and that $\|\Gamma f\|$ satisfies (3.8).

b) The proof of (3.9) follows as in a), with the only remark that instead of (4.37), we make use of the inequalities

$$\|\Gamma f_{n,i}(t)\| \leq \exp(\rho_\Gamma(\|\Lambda_1 f_{0,i}\|)t) \|\Gamma f_{0,i}\| \leq \exp(\rho_\Gamma(\|\Lambda_1 f_0\|)t) \|\Gamma f_0\| \quad (t \geq 0),$$

which are immediate by Lemma 4.4b), because ρ_Γ is non-decreasing. □

5. Applications to Kinetic Equations

Beside their intrinsic interest, the examples of this section illustrate how the abstract results of Section 3 can be put to work in various particular situations. The first example is a rather straightforward application of Theorem 3.1a), but represents a new approach to the existence theory for the Smoluchowski coagulation equation. The second example is an application of Corollary 3.1 to a space-dependent kinetic model. The third example is an application of Theorem 3.1a), which requires a more careful choice of the function a, and needs a more involved verification of the monotonicity properties required in (A_1) and (A_2).

Finally we mention that Theorem 3.1b) is equally applicable to the first and the third example, considered in this section. However, this does not bring any new relevance to our analysis, so that will be not discussed here.

5.1. **Smoluchowski's coagulation equation.** In this example, we reconsider the existence and uniqueness of strong, global in time solutions to the Smoluchowski's coagulation equation introduced in [10], [11] (see also, e.g., [18], for a recent review), to describe the irreversible evolution of particles that may coalesce into larger clusters. Let $f(t,y) \geq 0$ denote the density of clusters of size $y \in \mathbb{R}_+$ at time $t \geq 0$. Then the i.v.p. for the continuous version of the Smoluchowski's equation [11] reads

$$\frac{\partial}{\partial t} f = Q_c(f) = Q_c^+(f) - Q_c^-(f), \quad f(0) = f_0 \geq 0 \quad (t \geq 0), \tag{5.1}$$

with

$$Q_c^+(g)(y) := \frac{1}{2}\int_0^y q(y-y_*, y_*) g(y-y_*) g(y_*) dy_*, \tag{5.2}$$

$$Q_c^-(g)(y) := g(y) \int_0^\infty q(y, y_*) g(y_*) dy_*, \tag{5.3}$$

where the (coagulation) kernel $q : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a symmetric, measurable function.

We assume that there exist the constants $q_0, q_1 \geq 0$ and $0 \leq \alpha \leq \beta$, such that

$$q(y, y_*) \leq q_0 + q_1 (y^\alpha y_*^\beta + y^\beta y_*^\alpha) \quad (y, y_* \geq 0), \tag{5.4}$$

where

$$\alpha + \beta \leq 1. \tag{5.5}$$

Note that (5.4) includes the case when either $q_0 = 0$ or $q_1 = 0$. However, we suppose, without loss of generality, that $q_1 > 0$ (because the situation when q is bounded by a constant can be considered as a particularization of (5.4) to the case where $q_1 > 0$ and $\alpha = \beta = 0$).

For $k \geq 0$, let $L_k^1 := L_k^1(\mathbb{R}_+; dy)$ be the space of real measurable functions $g : \mathbb{R}_+ \mapsto \mathbb{R}$ such that

$$\|g\|_{L_k^1} := \int_{\mathbb{R}_+} (1+y)^k |g(y)| \, dy < \infty.$$

Denote $L_{k,+}^1 = \{g \in L_k^1 : g \geq 0\}$. The following property of the Smoluchowski's model is essential for our analysis. Suppose that the measurable functions $g, \psi : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfy $g, g\psi \in L_\beta^1$. Then

$$\int_0^\infty \psi(y) \left[Q_c^+(g)(y) - Q_c^-(g)(y)\right] dy = \frac{1}{2}\int_0^\infty \int_0^\infty \tilde{\psi}(y, y_*) q(y, y_*) g(y) g(y_*) dy dy_*, \tag{5.6}$$

where

$$\widetilde{\psi}(y, y_*) := \psi(y + y_*) - \psi(y) - \psi(y_*). \tag{5.7}$$

To obtain (5.6), simply apply the change of variables $(y, y_*) \to (y - y_*, y_*)$ in the first term of the l.h.s. of (5.6), and apply Fubini's theorem.

In particular, if $g \in L^1_{1+\beta}$, then taking $\psi(y) = y$ in identity (5.6), we get

$$\int_0^\infty Q_c(g)(y) y dy = 0. \tag{5.8}$$

This gives formally the mass conservation for Eq. (5.1).

Consider problem (5.1) in the space $X = L^1(\mathbb{R}_+; dy)$. Here $X = L^1(\mathbb{R}_+; dy)$ is equipped with the usual norm $\|\cdot\| = \|\cdot\|_{L^1}$, and with the natural order $\leq$. In this case, one can apply Theorem 3.1a) as follows:

Consider L^1_k as a subset of X. Let $i = 0, 1$ and define the positive linear operators $\Lambda_{c.i} : \mathcal{D}(\Lambda_{c.i}) \subset X \mapsto X$ by $\mathcal{D}(\Lambda_{c.i}) = L^1_{\gamma_i}$. $(\Lambda_{c.i} g)(y) := \lambda_i(y) g(y)$, with $\lambda_i(y) := (1 + y)^{\gamma_i}$, $y \geq 0$ a.e., where $\gamma_0 = \beta$ and $\gamma_1 = \alpha + \beta$.

Next note that formulae (5.2) and (5.3) define Q_c^+ and Q_c^- as positive and isotone nonlinear operators in X with the common domain $\mathcal{D}_c := L^1_\beta$.

It is immediate that $Q_c^\pm$, $\Lambda_{c.0}$, and $\Lambda_{c.1}$ satisfy the domain conditions for $Q^\pm$, Λ. and Λ_1, respectively, required by Theorem 3.1 a). In particular, the operators $Q_c^\pm$ are o-closed, by the monotone convergence theorem. Obviously, $\Lambda_{c.0}$ satisfies the conditions of (A_0) for Λ.

Now we check that $\Lambda_{c.i}$ $(i = 0.1)$ and $Q_c^\pm$ verify inequalities of the form (3.2) and (3.4). Indeed, if $g \in L^1_{2\beta,+}$, then starting from (5.6), we find

$$\|\Lambda_{c.i} Q_c^-(g)\| - \|\Lambda_{c.i} Q_c^+(g)\|$$

$$= \frac{1}{2} \int_0^\infty \int_0^\infty [(1 + y)^{\gamma_i} + (1 + y_*)^{\gamma_i} - (1 + y + y_*)^{\gamma_i}]\, q(y, y_*) g(y) g(y_*) dy dy_* \geq 0, \tag{5.9}$$

because $0 \leq \gamma_i \leq 1$, and

$$\frac{(1 + y)^\gamma + (1 + y_*)^\gamma}{(1 + y + y_*)^\gamma} \geq \inf_{x \geq 0} \frac{1 + x^\gamma}{(1 + x)^\gamma} = 1 \quad (0 \leq \gamma \leq 1, \quad y, y' \geq 0). \tag{5.10}$$

Inequality (5.9) shows that $g \mapsto \Delta_c(g) := \|\Lambda_{c.0} Q_c^-(g)\| - \|\Lambda_{c.0} Q_c^+(g)\|$ defines a positive isotone map $\Delta_c : \mathcal{D}(\Delta_c) \mapsto \mathbb{R}$ with domain $\mathcal{D}(\Delta_c) = L^1_{2\beta,+}$.

Starting again from (5.6), we find that if $g \in L^1_{3\beta,+}$, then

$$\|\Lambda^2_{c.0} Q_c^+(g)\| - \|\Lambda^2_{c.0} Q_c^-(g)\|$$

$$= \frac{1}{2} \int_0^\infty \int_0^\infty \left[(1 + y + y_*)^{2\beta} - (1 + y)^{2\beta} - (1 + y_*)^{2\beta}\right] q(y, y_*) g(y) g(y_*) dy dy_*. \tag{5.11}$$

If $0 \leq \beta \leq 1/2$, applying again (5.10) in (5.11), we get

$$\|\Lambda^2_{c.0} Q_c^+(g)\| - \|\Lambda^2_{c.0} Q_c^-(g)\| \leq 0, \tag{5.12}$$

which is of the form (3.4) with $\rho \equiv 0$.

On the other hand, if $1/2 < \beta \leq 1$, then to estimate (5.11), we apply the following form of Povzner's algebraic inequality ([3])

$$(1 + y + y_*)^{2\beta} - (1 + y)^{2\beta} - (1 + y_*)^{2\beta} \leq 2(1 + y)^\beta (1 + y_*)^\beta \quad (y, y_* \geq 0), \tag{5.13}$$

which can be easily proved: Indeed (5.13) is equivalent to $\zeta(x) = 2x^\beta + 1 + x^{2\beta} - (1+x)^{2\beta} \geq 0$ for all $x > 0$. However, as $\zeta(x^{-1}) = x^{-2\beta}\zeta(x)$, to prove that $\zeta(x) \geq 0$ for $x > 0$, we need only show that $\zeta(x) \geq 0$ on $(0,1]$, which is immediate, because $1/2 < \beta \leq 1$.

Thus, applying (5.13) in (5.11), we find that there is a number $c > 0$ such that

$$\left\|\Lambda^2_{c,0}Q^+_c(g)\right\| - \left\|\Lambda^2_{c,0}Q^-_c(g)\right\| \leq c\left\|\Lambda_{c,1}g\right\| \left\|\Lambda^2_{c,0}g\right\|. \tag{5.14}$$

Clearly, inequality (5.14) is of the form (3.4) with $\rho(x) = cx$.

Let $a_c(x) := a_0 x$, for some constant $a_0 > 0$. If a_0 is sufficiently large, then the map $L^1_{\beta,+} \ni g \mapsto a_0 \left\|\Lambda_{c,0}g\right\| \Lambda_{c,0}g - Q^-_c(g) \in X$ has the properties required in (A_2).

It appears that $Q^\pm_c$, $\Lambda_{c,0}$, $\Lambda_{c,1}$ and a_c verify the conditions of Theorem 3.1a) for $Q^\pm$, Λ, Λ_1 and a, respectively, provided that a_0 is sufficiently large. Consequently, one can apply Theorem 3.1a) to the i.v.p. (5.1). We obtain

Theorem 5.1. *Let $f_0 \in L^1_{2\beta,+}$ in Problem (5.1). Then Eq. (5.1) has a unique strong solution f such that $f(t) \in L^1_{2\beta,+}$, $t \geq 0$, and $\|f(t)\|_{L^1_{2\beta}}$ is locally bounded on $\mathbb{R}_+$. In addition $f, (1+y)^\beta f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+, dy))$,*

$$\|f(t)\|_{L^1_\beta} + \int_0^t \Delta_c(f(s))ds = \|f_0\|_{L^1_\beta} \quad (t \geq 0), \tag{5.15}$$

and there is a constant $c > 0$ such that

$$\|f(t)\|_{L^1_{2\beta}} \leq \exp(c\|f_0\|_{L^1_{\alpha+\beta}}\, t)\, \|f_0\|_{L^1_{2\beta}} \quad (t \geq 0). \tag{5.16}$$

It should be emphasized here that Theorem 5.1 does not imply directly the mass conservation, except for the case $q_1 > 0$, $\beta = 1$ and $\alpha = 0$. In fact, if $0 \leq 2\beta < 1$, then the theorem allows for the existence of solutions with infinite initial mass (see also [31]) i.e., $f_0 \in L^1_{2\beta,+}$, but $f_0 \notin L^1_1$.

However, if $f_0 \in L^1_{2\beta,+} \cap L^1_1$, then the solution $f(t)$ has finite mass. Indeed, let $\Gamma : L^1_1 \subset L^1 \mapsto L^1$ be defined by $(\Gamma g)(y) = yg(y)$ a.e. on $\mathbb{R}_+$. Clearly, Γ is of type D on $\cap_{k=1}^\infty L^1_{k\beta,+}$. Then proposition 3.1a) applies, so that $f \in L^1_{2\beta,+} \cap L^1_1$, and $\|\Gamma f(t)\| \leq \|\Gamma f_0\|$.

If L^1_r is replaced by $l^1_r(\mathbb{R}) = \{c = (c_j) : c_j \in \mathbb{R}, j = 1,2,..., \|c\|_r := \sum_{j=1}^\infty j^r |c_j| < \infty\}$, $r \geq 0$, then the above analysis remains valid, such that Theorem 5.1 can be reformulated, with obvious modifications, for the discrete version of the Smoluchowski equation [10]

$$\dot{c}_j = \frac{1}{2}\sum_{k=1}^{j-1} Q_{j-k,k}(c(t)) - \sum_{k=1}^{\infty} Q_{j,k}(c(t)), \quad c_j(0) = c_{j,0} \geq 0 \quad (j = 1,2,....), \tag{5.17}$$

where $Q_{j,k}(c) := q(k,j)c_k c_j$, is defined by the same symmetric coagulation kernel introduced before, subject to (5.4), (5.5). Here the component $c_j(t) \geq 0$ of $c(t) := (c_j(t))$ is interpreted as the concentration of clusters of size j at time $t \geq 0$.

Existence and uniqueness of global in time solutions to the Smoluchowski equation are rather well understood, in the context of coagulation-fragmentation theory, see. e.g., [18]-[32] (for more details, the interested reader is referred to [18], [31], and [32]). Existence and/or uniqueness of different kind of solutions was proved in several works, under various moment and regularity hypotheses. Thus, the results of [19]-[28] are valid for more or less particular variants of q given in (5.4). Moreover, in [29], it is assumed that $q(y, y_*) \leq \varphi(y)\varphi(y_*)$, with $\varphi(y) \leq k(1+y)^{\frac{1}{2}}$, for some constant $k > 0$. Furthermore, paper [30] considered a coagulation kernel that blows up for small values of y an y_*. Finally, in [31], existence of solutions was proved for q continuous, with $q(y, y_*)/(yy_*) \to 0$ as $(y, y_*) \to \infty$, while local existence and uniqueness of solutions was obtained under the condition $q(y, y_*) \leq \varphi(y)\varphi(y_*)$, with $\varphi(y)$ continuous and sublinear (i.e., $\varphi(\mu y) \leq \mu\varphi(y)$, $y \geq 0$, $\mu \geq 1$). The uniqueness results of [31] also include cases of solutions with infinite mass. The mass conservation in the context of the Smoluchowski equation has been recently investigated in [28] and [33].

Theorem 5.1 seems to be complementary to the literature, and to bring some unity into the existence theory for the strong solutions of Eqs.(5.1)-(5.3) and (5.17), under the general conditions (5.4) and (5.5).

5.2. **Povzner-like model with dissipative collisions.** We revise here the theory of existence and uniqueness of global solutions for a space-dependent Boltzmann model, [8] (see also [34]), with collision operators mollified in the spatial variable. The model describes a rarefied mono-component fluid of particles of unit mass, evolving in the free space with dissipative (conservative) binary collisions, i.e., collisions resulting in the loss (conservation) of the kinetic energy of the encounters.

According to the model [8], the post-collision velocities $\mathbf{v}'$, $\mathbf{w}'$ are related to the pre-collision velocities $\mathbf{v}$ and $\mathbf{w}$ by

$$\mathbf{v}' = \mathbf{v} - (1 - \beta(\mathbf{n}))\langle \mathbf{v} - \mathbf{w}, \mathbf{n}\rangle\mathbf{n}, \quad \mathbf{w}' = \mathbf{w} + (1 - \beta(\mathbf{n}))\langle \mathbf{v} - \mathbf{w}, \mathbf{n}\rangle\mathbf{n}, \tag{5.18}$$

where $< \cdot, \cdot >$ is the inner product in $\mathbb{R}^3$ and $\mathbf{n} \in \mathbb{S}^2$ -the unit sphere in $\mathbb{R}^3$. Here, $\beta : \mathbb{S}^2 \mapsto [0, 1/2)$ is a given measurable function. The total momentum is conserved in collisions, $\mathbf{v}' + \mathbf{w}' = \mathbf{v} + \mathbf{w}$, but the kinetic energy is lost

$$|\mathbf{v}'|^2 + |\mathbf{w}'|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2\beta(\mathbf{n})(1 - \beta(\mathbf{n}))\,|\langle \mathbf{v} - \mathbf{w}, \mathbf{n}\rangle|^2, \tag{5.19}$$

excepting the case $\beta = 0$, when the collisions become elastic.

For each fixed $\mathbf{n} \in \mathbb{S}^2$, the transformation $\mathbb{R}^3 \times \mathbb{R}^3 \ni (\mathbf{v}, \mathbf{w}) \mapsto (\mathbf{v}', \mathbf{w}') \in \mathbb{R}^3 \times \mathbb{R}^3$ is invertible. The inversion formulae are

$$\hat{\mathbf{v}} = \mathbf{v} - \left(\frac{1 - \beta(\mathbf{n})}{1 - 2\beta(\mathbf{n})}\right)\langle \mathbf{v} - \mathbf{w}, \mathbf{n}\rangle\mathbf{n}, \quad \hat{\mathbf{w}} = \mathbf{w} + \left(\frac{1 - \beta(\mathbf{n})}{1 - 2\beta(\mathbf{n})}\right)\langle \mathbf{v} - \mathbf{w}, \mathbf{n}\rangle\mathbf{n}.$$

Let $X = L^1(\mathbb{R}^3 \times \mathbb{R}^3; d\mathbf{x}d\mathbf{v}) = L^1$, equipped with the norm $\|\cdot\| := \|\cdot\|_{L^1}$ and the natural order $\leq$. Define by $L^1_k := L^1_k(\mathbb{R}^3 \times \mathbb{R}^3; d\mathbf{x}d\mathbf{v})$, $k \in \mathbb{R}$, the space of measurable functions on $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ satisfying

$$\|g\|_{L^1_k} := \int_{\mathbb{R}_+} (1 + |\mathbf{v}|^2)^{\frac{k}{2}}\, |g(\mathbf{x}, \mathbf{v})|\, d\mathbf{x}d\mathbf{v} < \infty.$$

As before, $L^1_{k,+}$ denotes the positive cone in L^1_k.

Formulated in X, the i.v.p. for the above model reads

$$\frac{d}{dt}f = Af + Q^+_d(f) - Q^-_d(f), \quad f(0) = f_0 \geq 0, \tag{5.20}$$

where $f = f(t, \mathbf{x}, \mathbf{v})$ is the one-particle distribution function, depending on time $t \geq 0$, position $\mathbf{x} \in \mathbb{R}^3$, and velocity $\mathbf{v} \in \mathbb{R}^3$ of the so-called test particle, A is the infinitesimal generator of the C_0 group $\{U^t\}_{t\in\mathbb{R}}$ of the free motion $(U^t f)(\mathbf{x}, \mathbf{v}) := f(\mathbf{x} - t\mathbf{v}, \mathbf{v})$, a.e. Further, Q^+_d and Q^-_d are the so-called nonlinear gain and loss operators, respectively, and describe the influence of the collisions on the evolution of f. They are formally defined by

$$Q_d^+(g)(\mathbf{x},\mathbf{v})$$
$$:= \int_0^R dr \int_{\mathbb{S}^2\times\mathbb{R}^3} \frac{1}{(1-2\beta(\mathbf{n}))^{1+\gamma}} |\langle \mathbf{n}, \mathbf{v}-\mathbf{w}\rangle|^\gamma P(r,\mathbf{n}) g(\mathbf{x},\hat{\mathbf{v}}) g(\mathbf{x}+r\mathbf{n},\hat{\mathbf{w}}) d\mathbf{n}\, d\mathbf{w}, \tag{5.21}$$

and

$$Q_d^-(g)(\mathbf{x},\mathbf{v}) := g(\mathbf{x},\mathbf{v}) \int_0^R dr \int_{\mathbb{S}^2\times\mathbb{R}^3} |\langle \mathbf{n}, \mathbf{v}-\mathbf{w}\rangle|^\gamma P(r,\mathbf{n}) g(\mathbf{x}+r\mathbf{n},\mathbf{w}) d\mathbf{n}\, d\mathbf{w}, \tag{5.22}$$

respectively, where $P : \mathbb{R}_+ \times \mathbb{S}^2 \mapsto \mathbb{R}_+$ is a given measurable function with $P(r,\mathbf{n}) = P(r,-\mathbf{n})$ assumed to satisfy

$$P(r,\mathbf{n}) \le c_0 r^2 \quad (r \ge 0,\ \mathbf{n} \in \mathbb{S}^2), \tag{5.23}$$

for some constants $c_0 > 0$, $0 \le \gamma \le 1$, and $R > 0$, specific to the collision processes.

Under the above assumptions, formulae (5.21) and (5.22) define $Q_d^\pm$ as positive and isotone operators on the common domain $\mathcal{D} := L^1_\gamma$. This follows easily if we perform the change of variable $(0,R] \times \mathbb{S}^2 \ni (r,\mathbf{n}) \mapsto \mathbf{y} := r\mathbf{n} \in \{\mathbf{z} \in \mathbb{R}^3 : |\mathbf{z}| \le R\}$ in (5.21) and (5.22), and then take into account (5.23).

The basic property of the model is the identity

$$\int_{\mathbb{R}^3} \psi(\mathbf{v}) \left[Q_d^+(g) - Q_d^-(g)\right] d\mathbf{v}$$
$$= \frac{1}{2} \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} \tilde{\psi}(\mathbf{v},\mathbf{w},\mathbf{v}',\mathbf{w}') |\langle \mathbf{n}, \mathbf{w}-\mathbf{v}\rangle|^\gamma P(r,\mathbf{n}) g(\mathbf{x},\mathbf{v}) g(\mathbf{x}+r\mathbf{n},\mathbf{w}) d\mathbf{n} d\mathbf{v} d\mathbf{w}, \tag{5.24}$$

where $\psi, g : \mathbb{R}^3 \mapsto \mathbb{R}$ are measurable functions such that $\psi g \in L^1_\gamma$, and

$$\tilde{\psi}(\mathbf{v},\mathbf{w},\mathbf{v}',\mathbf{w}') := \psi(\mathbf{v}') + \psi(\mathbf{w}') - \psi(\mathbf{v}) - \psi(\mathbf{w}),$$

with $\mathbf{v}'$ and $\mathbf{w}'$ given by (5.18). We deduce easily (5.24), performing the change of variable $(v,w) \to (\hat{v},\hat{w})$ in the first term of the l.h.s (5.24).

If $\beta \equiv 0$, then (5.20) yields a version of the so-called generalized Boltzmann equation with binary elastic (conservative) collisions, analyzed in [9].

Let the positive linear operator $\Lambda_d : L^1_2 \mapsto X$ be defined by $(\Lambda_d g)(\mathbf{x},\mathbf{v}) := \lambda(\mathbf{v}) g(\mathbf{x},\mathbf{v})$ a.e. on $\mathbb{R}^3 \times \mathbb{R}^3$, with $\lambda(\mathbf{v}) := (1+|\mathbf{v}|^2)$. Define $a_d(x) := c_0 x$ for some constant $c_0 > 0$. If c_0 is sufficiently large, then a_d, Λ_d and $Q_d^\pm$ verify the conditions of Corollary 3.1 for a, $\Lambda = \Lambda_1$ and $Q^\pm$, respectively.

Indeed, the operators $Q_d^\pm$ are p-saturated. Moreover, they are o-closed, by the monotone convergence theorem. It is immediate that the domain conditions imposed in Sections 1, 3 and in Corollary 3.1 are satisfied. Further, applying (5.19) in (5.24), we obtain an inequality of the form (3.2), i.e., if $g \in L^1_{4,+}$, then

$$0 \le \Delta_d(g) := \|\Lambda_d Q_d^-(g)\| - \|\Lambda_d Q_d^+(g)\|$$
$$= \int_0^R dr \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \pi(r,\mathbf{n},\mathbf{v},\mathbf{w},\mathbf{x}) g(\mathbf{x},\mathbf{v}) g(\mathbf{x}+r\mathbf{n},\mathbf{w}) d\mathbf{n} d\mathbf{v} d\mathbf{w} d\mathbf{x},$$

where $\pi(r,\mathbf{n},\mathbf{v},\mathbf{w},\mathbf{x}) := \beta(\mathbf{n})(1-\beta(\mathbf{n})) |\langle \mathbf{n}, \mathbf{v}-\mathbf{w}\rangle|^{2+\gamma} P(r,\mathbf{n})$. Remark here that the map $L^1_{4,+} \ni g \mapsto \Delta_d(g) \in \mathbb{R}$ is positive and isotone. Moreover, for c_0 sufficiently large, the map $L^1_{2,+} \ni g \mapsto c_0 \|\Lambda_d g\| \Lambda_d g - Q^-_d(g) \in X$ is also positive and isotone. Further, to obtain an inequality of the form (3.4), note that (5.19) gives $\lambda(\mathrm{v}')^2 + \lambda(\mathrm{w}')^2 \leq (2+|\mathrm{v}'|^2+|\mathrm{w}'|^2)^2 \leq (2+|\mathrm{v}|^2+|\mathrm{w}|^2)^2 = \lambda(\mathrm{v})^2+\lambda(\mathrm{w})^2+2\lambda(\mathrm{v})\lambda(\mathrm{w})$, which can be applied in (5.24) to conclude easily that there are two constants c_1, $c > 0$ such that

$$\left\|\Lambda_d^2 Q_c^+(g)\right\| - \left\|\Lambda_d^2 Q_d^-(g)\right\|$$

$$\leq c_1 \int_0^R dr \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} r^2\lambda(\mathrm{v})\lambda(\mathrm{w})^{1+\frac{\gamma}{2}} g(\mathrm{x},\mathrm{v})g(\mathrm{x}+r\mathrm{n},\mathrm{w})d\mathrm{n}d\mathrm{v}d\mathrm{w}d\mathrm{x}$$

$$\leq c\left\|\Lambda_d g\right\| \left\|\Lambda_d^2 g\right\|,$$

for all $g \in L^1_{6,+}$. Finally, it is obvious that the group U^t (generated by A) commutes with the semigroup V^t (generated by Λ_d), and $\Lambda^k Q^+(U^{\cdot} g) \in L^1_{loc}(\mathbb{R}_+; X_+)$ for all $g \in \cap_{n=1}^{\infty} L^1_{n,+}$, $k = 1, 2, \ldots$.

Therefore, by Corollary 3.1, we have the following result:

Theorem 5.2. *Let $f_0 \in L^1_{4,+}$ in Problem (5.20). Then Eq. (5.20) has a unique positive mild solution f such that $f(t) \in L^1_{4,+}$, $t \geq 0$, and $\|f(t)\|_{L^1_4}$ is locally bounded on $\mathbb{R}_+$. In addition, f, $(1+|\mathrm{v}|^2)f \in C(\mathbb{R}_+; L^1)$,*

$$\|f(t)\|_{L^1_2} + \int_0^t \Delta_d(f(s))ds = \|f_0\|_{L^1_2} \quad (t \geq 0), \tag{5.25}$$

and there is a constant $c > 0$ such that

$$\|f(t)\|_{L^1_4} \leq \exp(c\|f_0\|_{L^1_2} t)\,\|f_0\|_{L^1_4} \quad (t \geq 0). \tag{5.26}$$

It is not difficult to see that the argument of Theorem 5.2 can be repeated with obvious modifications to provide a similar result for the space-homogeneous version of Eqs. (5.20)-(5.22), which coincides with the force-free, three-dimensional space-homogeneous Boltzmann model for granular flows [13], [14].

Theorem 5.2 improves a result announced in [8], where the existence of solutions was stated for initial data in L^1_6, and property (5.25) was included among the conditions for uniqueness of the solution.

5.3. **Boltzmann model with inelastic collisions and chemical reactions.** In this final example, we reconsider the existence theory of solutions for an abstract system of a Boltzmann-like phenomenological equations, [6], [35], [36], which describe a multi-component reacting gas of particles with internal states, characterized by discrete values of the internal energy. Motivated by the fact that a real gas mixture of particles with internal structure can be thought as a mixture of several chemical species of mass points with unique internal state, any gas particle of the model is supposed to posses one internal state. Specifically, the model refers to a gas consisting of N chemical species. A particle of species $n = 1, 2, \ldots, N$ is characterized by its mass $m_n > 0$ and internal energy E_n. Without loss of generality, one can assume that $E_n \geq 0$, $1 \leq n \leq N$. It is assumed that the chemical reactions are induced by inelastic (possibly) multi-body, instant collisions. A reaction is identified with a couple $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$, where $\mathcal{M} := \{\gamma = (\gamma_n)_{1\leq n\leq N} \mid \gamma_n \in \{0, 1, \ldots, K\}\}$ is a multi-index set. Here $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathcal{M}$ and $\beta = (\beta_1, \ldots, \beta_N) \in \mathcal{M}$ designate the pre-collision and post-collision channels, respectively, with $0 \leq \alpha_n, \beta_n \leq K$ participants of species n; $1 \leq n \leq N$. Any couple of the form $(\gamma, \gamma) \in \mathcal{M} \times \mathcal{M}$ is identified with a multi-body elastic collision with γ_n collision partners of species n; $1 \le n \le N$. The number of particles in some channel $\gamma \in \mathcal{M}$ is $|\gamma| := \sum_{i=1}^{N} \gamma_i$. The family of chemical species participating in channel γ is denoted by $\mathcal{N}(\gamma) := \{i : \gamma_i > 0, 1 \le i \le N\}$.

Let M_γ, $\mathbf{V}_\gamma(\mathbf{w})$ and $W_\gamma(\mathbf{w})$ denote the total mass, velocity of the mass center and total energy, respectively, for the particles in channel γ, i.e.,

$$M_\gamma := \sum_{i=1}^{N} \gamma_i m_i, \tag{5.27}$$

$$\mathbf{V}_\gamma(\mathbf{w}) := \frac{1}{M_\gamma} \sum_{i \in \mathcal{N}(\gamma)} \sum_{j=1}^{\gamma_i} m_i \mathbf{w}_{i,j}, \tag{5.28}$$

$$W_\gamma(\mathbf{w}) := \sum_{i \in \mathcal{N}(\gamma)} \sum_{j=1}^{\gamma_i} (2^{-1} m_i \mathbf{w}_{i,j}^2 + E_i), \tag{5.29}$$

where $\mathbf{w} = ((\mathbf{w}_{k,i})_{i \in \{1,\dots,\alpha_k\}})_{k \in \mathcal{N}(\gamma)}$ represents the ensemble of velocities of the particles in channel γ. Then, the kinetic energy of the particles (with velocities $\mathbf{w}$) in channel γ, relative to the frame of the mass center, reads

$$W_{r,\gamma}(\mathbf{w}) = W_\gamma(\mathbf{w}) - \frac{M_\gamma \mathbf{V}_\gamma(\mathbf{w})^2}{2} - \sum_{i=1}^{N} \gamma_i E_i. \tag{5.30}$$

Obviously, $W_{r,\gamma}(\mathbf{w}) \ge 0$.

According to the model, a gas reaction (α, β) may take place only if it is consistent with the conservation of mass, momentum and energy, i.e.,

$$M_\alpha = M_\beta, \quad \mathbf{V}_\alpha(\mathbf{w}) = \mathbf{V}_\beta(\mathbf{u}), \quad W_\alpha(\mathbf{w}) = W_\beta(\mathbf{u}). \tag{5.31}$$

We will assume here that elastic collisions are always present in the gas evolution. Therefore, the set $\mathcal{C}_M := \{(\alpha, \beta) \in \mathcal{M} \times \mathcal{M} : M_\alpha = M_\beta\}$ is nonempty.

The Boltzmann-like system of equations for the above model is

$$\frac{\partial}{\partial t} f_i = Q_i^+(f) - Q_i^-(f) \qquad (t \ge 0, \quad 1 \le i \le N). \tag{5.32}$$

Here the unknown $f_i : \mathbb{R}_+ \times \mathbb{R}^3 \mapsto \mathbb{R}_+$ is the one particle distribution functions $f_i = f_i(t, \mathbf{v})$ (t-time, $\mathbf{v}$-velocity) of the particles of species $1 \le i \le N$. In Eq. (5.32), $Q_i^+(f)$ and $Q_i^-(f)$, with $f := (f_1, \dots, f_N)$, are the so-called loss and gain (nonlinear) operators for the particles of species i, respectively. Formally,

$$Q_i^+(g)(\mathbf{v}) = \sum_{\alpha,\beta \in \mathcal{M}} \alpha_i \int_{\mathbb{R}^{3|\alpha|-3} \times \Omega_\beta} [p_{\beta,\alpha}(\mathbf{w}, \mathbf{n})(g^\beta \circ \mathbf{u}_{\beta,\alpha})(\mathbf{w}, \mathbf{n})]_{\mathbf{w}_{i,\alpha_i} = \mathbf{v}} \, d\tilde{\mathbf{w}}_i d\mathbf{n}, \tag{5.33}$$

$$Q_i^-(g)(\mathbf{v}) = \sum_{\alpha,\beta \in \mathcal{M}} \alpha_i \int_{\mathbb{R}^{3|\alpha|-3} \times \Omega_\beta} [r_{\beta,\alpha}(\mathbf{w}, \mathbf{n}) g^\alpha(\mathbf{w})]_{\mathbf{w}_{i,\alpha_i} = \mathbf{v}} \, d\tilde{\mathbf{w}}_i d\mathbf{n}, \tag{5.34}$$

where

$$g^\gamma(\mathbf{w}) := \prod_{i \in \mathcal{N}(\gamma)} \prod_{j=1}^{\gamma_i} g_i(\mathbf{w}_{i,j}), \quad \gamma \in \mathcal{M}. \tag{5.35}$$

Ω_γ is the unit sphere in $\mathbb{R}^{3|\gamma|-3}$, with $\gamma \in \mathcal{M}$, and $d\tilde{\mathbf{w}}_i$ is the Euclidean element of area on $\{\mathbf{w} \in \mathbb{R}^{3|\alpha|} \mid \mathbf{w}_{i,\alpha_i} = \mathbf{v}\}$. Here, the functions $\mathbf{u}_{\beta,\alpha} \in C(\mathbb{R}^{3|\alpha|} \times \Omega_\beta; \mathbb{R}^{3|\beta|})$, and the measurable functions $r_{\beta,\alpha}, p_{\beta,\alpha} : \mathbb{R}^{3|\alpha|} \times \Omega_\beta \mapsto \mathbb{R}_+$ are given.

The following conditions are assumed ([6], [36], [37]):

(B_1) $r_{\beta,\alpha} = p_{\beta,\alpha} = 0$ unless: $|\alpha| \geq 2$, $|\beta| \geq 2$, $(\alpha,\beta) \in \mathcal{C}_M$, and $\mathbf{w} \in D^+_{\beta,\alpha} := \left\{\mathbf{w}' \in \mathbb{R}^{3|\alpha|} : W_{r,\alpha}(\mathbf{w}') + \sum_{i=1}^{N}(\alpha_i - \beta_i)E_i \geq 0\right\}$.

(B_2) For each $i \in \mathcal{N}(\alpha)$ fixed, $p_{\beta,\alpha}(\mathbf{w},\mathbf{n})$, $r_{\beta,\alpha}(\mathbf{w},\mathbf{n})$, and $u_{\beta,\alpha}(\mathbf{w})$ are invariant with respect to the interchange of the components $\mathbf{w}_{i,1}, ..., \mathbf{w}_{i,\alpha_i}$ of $\mathbf{w}$.

(B_3) If $(\alpha,\beta) \in \mathcal{C}_M$, $\mathbf{w} \in D^+_{\beta,\alpha}$, then

$$(V_\beta \circ \mathbf{u}_{\beta,\alpha})(\mathbf{w},\mathbf{n}) = V_\alpha(\mathbf{w}), \quad (W_\beta \circ \mathbf{u}_{\beta,\alpha})(\mathbf{w},\mathbf{n}) = W_\alpha(\mathbf{w}), \tag{5.36}$$

for all $\mathbf{n} \in \Omega_\beta$, and

$$\int_{\mathbb{R}^{3|\alpha|}\times\Omega_\beta} p_{\beta,\alpha}(\mathbf{w},\mathbf{n})\varphi(\mathbf{w},\mathbf{n})(\psi \circ \mathbf{u}_{\beta,\alpha})(\mathbf{w},\mathbf{n})d\mathbf{w}d\mathbf{n}$$

$$= \int_{\mathbb{R}^{3|\beta|}\times\Omega_\alpha} r_{\alpha,\beta}(\mathbf{w},\mathbf{n})(\varphi \circ \mathbf{u}_{\alpha,\beta})(\mathbf{w},\mathbf{n})\psi(\mathbf{w},\mathbf{n})d\mathbf{w}d\mathbf{n}, \tag{5.37}$$

for all $\varphi : \mathbb{R}^{3|\alpha|} \mapsto \mathbb{R}$ and $\psi : \mathbb{R}^{3|\beta|} \mapsto \mathbb{R}$, for which the integrals are well defined.

We suppose that the reactions are reversible, i.e., if $r_{\beta,\alpha} \neq 0$ for some (α,β), then also $r_{\alpha,\beta} \neq 0$.

It follows from (5.37) that $p_{\beta,\alpha}$ and $r_{\beta,\alpha}$ are related one to another. Indeed, a more explicit relationship between $p_{\beta,\alpha}$ and $r_{\beta,\alpha}$ can be derived, as it results from a general example constructed in [36], [37], but this is beyond our present scope. Here we only mention that Eq. (5.32) reduces to the classical Boltzmann equation, when one assumes a mono-component gas of particles with binary elastic collisions (i.e., $N = 1$, $K = 2$, and $p_{\beta,\alpha} = r_{\beta,\alpha} = 0$ unless $\alpha = \beta = (1,1)$).

The last condition of the model concerns the behaviour of $r_{\beta,\alpha}$ (see [6]):

Assumption 5.1. *There are some constants* $0 \leq q \leq 1$ *and* $c_q > 0$ *such that*

$$\nu_{\beta,\alpha}(\mathbf{w}) := \int_{\Omega_\beta} r_{\beta,\alpha}(\mathbf{w},\mathbf{n})d\mathbf{n} \leq c_q(1 + W_\alpha(\mathbf{w}))^q \quad (\mathbf{w} \in \mathbb{R}^{|\alpha|}, \ a.e.). \tag{5.38}$$

for all $\alpha, \beta \in \mathcal{M}$.

Obviously, $\nu_{\beta,\alpha}(\mathbf{w}) = 0$, unless $(\alpha,\beta) \in \mathcal{C}_M$.

A consequence of (B_1), (B_2) and (5.38) is the key equality

$$\sum_{i=1}^{N} \int_{\mathbb{R}^3} \Psi_i^{(j)}(\mathbf{v}) \left[Q_i^+(g)(\mathbf{v}) - Q_i^-(g)(\mathbf{v})\right] d\mathbf{v} = 0 \quad (0 \leq j \leq 4), \tag{5.39}$$

for all $g = (g_1, ..., g_N)$ with $(1+ \mid \mathbf{v} \mid^2)^{1+q} g_i \in L^1(\mathbb{R}^3; d\mathbf{v})$, $i = 1, 2, ..., N$. Here,

$$\Psi_i^{(0)}(\mathbf{v}) := m_i, \quad \Psi_i^{(4)}(\mathbf{v}) := \frac{1}{2}m_i\,|\mathbf{v}|^2 + E_i, \quad \Psi_i^{(j)}(\mathbf{v}) := m_i v_j \quad (1 \leq i \leq N),$$

where v_j is the j-component, $j = 1,2,3$, of $\mathbf{v}$. Equality (5.39) implies, at lest formally, the bulk conservation of mass, momentum and total energy.

Let $X := (L^1(\mathbb{R}^3; d\mathbf{v}))^N$ be equipped with the order $\leq$ induced by the order of the components (i.e., the natural order of L^1). The norm on X is defined as

$$\|g\| := \sum_{i=1}^{N} \int_{\mathbb{R}^3} |g_i(\mathbf{v})|\, d\mathbf{v} = \sum_{i=1}^{N} \|g_i\|_{L^1}\,. \tag{5.40}$$

Denote by $L^1_k := L^1_k(\mathbb{R}^3; d\mathbf{v})$, $k \in \mathbb{R}$, the space of measurable functions $g : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ satisfying

$$\|g\|_{L^1_k} := \int_{\mathbb{R}_+} (1 + |\mathbf{v}|^2)^{\frac{k}{2}} |g(\mathbf{v})| \, d\mathbf{v} < \infty$$

and let $L^1_{k,+}$ be the positive cone in L^1_k.

It is natural to formulate the i.v.p. for Eq. (5.32) in the space X. Under the above conditions, formulae (5.33) and (5.34) define Q_i^+ and Q_i^-, $1 \le i \le N$, as operators from the common domain $\mathcal{D} = (L^1_2)^N \subset X$ to $L^1(\mathbb{R}^3; d\mathbf{v})$. Defining the operators $Q_B^{\pm} : \mathcal{D} \subset X \mapsto X$ by $Q_B^{\pm} = (Q_1^{\pm}, \dots\dots, Q_N^{\pm})$, we can write the i.v.p. for Eq. (5.32) in X as

$$\frac{d}{dt} f = Q_B^+(f) - Q_B^-(f), \quad f(0) = f_0 = (f_{0,1}, \dots, f_{0,N}) \in X_+. \tag{5.41}$$

We shall prove the existence of solutions to Problem (5.41), by applying Theorem 3.1a) (in the case $\Lambda = \Lambda_1$). To this end, let the positive closed linear operator $\Lambda_B : (L^1_2)^N \mapsto X$ be defined on components by $(\Lambda_B g)_i(\mathbf{v}) = \lambda_i(\mathbf{v}) g(\mathbf{v})$ a.e. on $\mathbb{R}^3 \times \mathbb{R}^3$, where $\lambda_i(\mathbf{v}) := m_i + m_i |\mathbf{v}|^2/2 + E_i$, $1 \le i \le N$. Denote $l_\gamma(\mathbf{w}) := \sum_{i \in \mathcal{N}(\gamma)} \sum_{j=1}^{\gamma_i} \lambda_i(\mathbf{w}_{i,j})$; $\gamma \in \mathcal{M}$. Then clearly, $l_\gamma(\mathbf{w}) = M_\gamma + W_\gamma(\mathbf{w})$, hence

$$0 \le W_\gamma(\mathbf{w}) < l_\gamma(\mathbf{w}). \tag{5.42}$$

In addition, defining $\lambda^\gamma(\mathbf{w}) := \prod_{i \in \mathcal{N}(\gamma)} \prod_{j=1}^{\gamma_i} \lambda_i(\mathbf{w}_{i,j})$, $\gamma \in \mathcal{M}$, we have

$$l_\gamma(\mathbf{w}) \le |\gamma| \, E^{1-|\gamma|} \lambda^\gamma(\mathbf{w}). \tag{5.43}$$

where $E := \min\{m_i + E_i : 1 \le i \le N\}$. It is useful to remark that, since $W_\gamma(\mathbf{w}) \ge E \mid \gamma \mid > 0$, and $0 \le q \le 1$, then by (5.38), (5.42) and (5.43),

$$\nu_{\beta,\alpha}(\mathbf{w}) \le C \lambda^\alpha(\mathbf{w}) \quad (\mathbf{w} \in \mathbb{R}^{|\alpha|}, \ a.e.). \tag{5.44}$$

for all $\alpha, \beta \in \mathcal{M}$. Here $C = C(E, K) > 0$ is a number depending on E and K (recall that K is the maximum number of partners in a reaction channel).

To apply Theorem 3.1a) to (5.41), first remark that $Q_B^{\pm}$ and Λ_B verify the domain conditions imposed to $Q^{\pm}$ and Λ by the theorem. Moreover, Λ_B has the properties required for Λ in (A_0). Further, observe that formula (5.39) provides correspondent to (3.2), specifically,

$$\Delta_B(g) := \|\Lambda_B Q_B^-(g)\| - \|\Lambda_B Q_B^+(g)\| = 0 \quad (g \in (L^1_{4,+})^N).$$

To obtain a correspondent to (3.4), define $s_\gamma(\mathbf{w}) := \sum_{i \in \mathcal{N}(\gamma)} \sum_{j=1}^{\gamma_i} \lambda_i(\mathbf{w}_{i,j})^2$. Next, using the definition of Q_B^+ and property (B_2), and applying the obvious inequality $s_\alpha(\mathbf{w}) \le l_\alpha(\mathbf{w})^2$, we find that if $g \in (L^1_{6,+})^N$, then

$$\|\Lambda_B^2 Q_B^+(g)\| = \sum_{\alpha,\beta \in \mathcal{M}} \int_{\mathbb{R}^{3|\alpha|} \times \Omega_\beta} s_\alpha(\mathbf{w}) p_{\beta,\alpha}(\mathbf{w}, \mathbf{n}) (g^\beta \circ u_{\beta,\alpha})(\mathbf{w}, \mathbf{n}) d\mathbf{w} d\mathbf{n}$$

$$\le \sum_{\alpha,\beta \in \mathcal{M}} \int_{\mathbb{R}^{3|\alpha|} \times \Omega_\beta} l_\alpha(\mathbf{w})^2 p_{\beta,\alpha}(\mathbf{w}, \mathbf{n}) (g^\beta \circ u_{\beta,\alpha})(\mathbf{w}, \mathbf{n}) d\mathbf{w} d\mathbf{n}.$$

We apply property (5.37) in the last integral. Then interchanging α and β, we get

$$\|\Lambda_B^2 Q_B^+(g)\| \le \sum_{\alpha,\beta \in \mathcal{M}} \int_{\mathbb{R}^{3|\alpha|} \times \Omega_\beta} (l_\beta \circ u_{\beta,\alpha})^2(\mathbf{w}, \mathbf{n}) r_{\beta,\alpha}(\mathbf{w}, \mathbf{n}) g^\alpha(\mathbf{w}) d\mathbf{w} d\mathbf{n}. \tag{5.45}$$

Since $l_\beta(\mathbf{w}) = M_\beta + W_\beta(\mathbf{w})$, property (B_3) implies that $(l_\beta \circ u_{\beta,\alpha})(\mathbf{w},\mathbf{n}) = l_\alpha(\mathbf{w})$ for all $(\alpha,\beta) \in \mathcal{C}_M$, $\mathbf{w} \in D^+_{\beta,\alpha}$. This and (B_1) enable us to deduce from (5.45) that

$$\left\|\Lambda_B^2 Q_B^+(g)\right\| \le \sum_{\alpha,\beta\in\mathcal{M}} \int_{\mathbb{R}^{3|\alpha|}\times\Omega_\beta} l_\alpha(\mathbf{w})^2 r_{\beta,\alpha}(\mathbf{w},\mathbf{n}) g^\alpha(\mathbf{w}) d\mathbf{w} d\mathbf{n}. \tag{5.46}$$

Now, using the definitions of $l_\alpha(\mathbf{w})$ and Q_B^-, and then, taking advantage of (5.38) and (5.42), we obtain from (5.46)

$$\left\|\Lambda_B^2 Q_B^+(g)\right\| \le \sum_{\alpha,\beta\in\mathcal{M}} \int_{\mathbb{R}^{3|\alpha|}\times\Omega_\beta} s_\alpha(\mathbf{w}) r_{\beta,\alpha}(\mathbf{w},\mathbf{n}) g^\alpha(\mathbf{w}) d\mathbf{w} d\mathbf{n} + \rho_B(\|\Lambda_B g\|)\left\|\Lambda_B^2 g\right\|$$

$$= \left\|\Lambda_B^2 Q_B^-(g)\right\| + \rho_B(\|(\Lambda_B g\|)\left\|\Lambda_B^2 g\right\|,$$

where ρ_B is a positive non-decreasing (polynomial) function.

Therefore, the last inequality is the required correspondent to (3.4) (in the case $\Lambda = \Lambda_1$).

Further, let $a_0 > 0$ be some constant, and define $a(x) := a_0 \sum_{p=1}^{NK} x^p$, $x \ge 0$. Therefore, $a(\|\Lambda_B g\|) = a_0 \sum_{p=1}^{NK} \|\Lambda_B g\|^p$. But, each term $\|\Lambda_B g\|^p$ in the r.h.s of the last equality can be expressed by (5.40), and the resulting expression can be expanded by the multinomial formula. Then, after some elementary algebra we get the following useful expression

$$a(\|\Lambda_B g\|) = a_0 \sum_{\gamma\in\mathcal{M},\,|\gamma|\ge 1} c_{\gamma,i} \int_{\mathbb{R}^{3|\gamma|}} \lambda^\gamma(\mathbf{w}) g^\gamma(\mathbf{w}) d\mathbf{w}, \tag{5.47}$$

where $c_{\gamma,i} > 0$ are strictly positive, constant coefficients, $\gamma \in \mathcal{M}$, $|\gamma| \ge 1$, $1 \le i \le N$.

We show that if a_0 is sufficiently large, then $(L^1_{2,+})^N \ni g \mapsto a(\|\Lambda_B f\|)\Lambda_B g - Q_B^-(g) \in X$ is positive and isotone. To this end, first note that one can write

$$Q_i^-(g)(\mathbf{v}) = R_i(g)(\mathbf{v})\, g_i(\mathbf{v}), \quad (g \in (L^1_{2,+})^N,\ \mathbf{v} \in \mathbb{R}^3\ a.e.,\ 1 \le i \le N),$$

where

$$R_i(g)(\mathbf{v}) := \sum_{\alpha,\beta\in\mathcal{M}} \alpha_i \int_{\mathbb{R}^{3|\alpha|-3}} \left[\nu_{\beta,\alpha}(\mathbf{w}) \prod_{\substack{s\in\mathcal{N}(\alpha)\\ (s,j)\ne(i,\alpha_i)}} \prod_{j=1}^{\alpha_s} g_s(\mathbf{w}_{s,j}) \right]_{\mathbf{w}_{i,\alpha_i}=\mathbf{v}} d\tilde{\mathbf{w}}_i, \tag{5.48}$$

with $\nu_{\beta,\alpha}$ as in (5.38). Hence,

$$a(\|\Lambda_B g\|)(\Lambda_B g)_i(\mathbf{v}) - Q_i^-(g)(\mathbf{v}) = [a(\|\Lambda_B g\|)\lambda_i(\mathbf{v}) - R_i(g)(\mathbf{v})]\, g_i(\mathbf{v}). \tag{5.49}$$

It is convenient to set

$$R_i^\Lambda(g)(\mathbf{v}) := C \sum_{\alpha,\beta\in\mathcal{M}} \alpha_i \int_{\mathbb{R}^{3|\alpha|-3}} \left[\lambda^\alpha(\mathbf{w}) \prod_{\substack{s\in\mathcal{N}(\alpha)\\ (s,j)\ne(i,\alpha_i)}} \prod_{j=1}^{\alpha_s} g_s(\mathbf{w}_{s,j}) \right]_{\mathbf{w}_{i,\alpha_i}=\mathbf{v}} d\tilde{\mathbf{w}}_i, \tag{5.50}$$

with C as in (5.44). Summing on β in (5.50), using the explicit form of $\lambda^\alpha(\mathbf{w})$, and invoking property (B_1), we are easily led to

$$R_i^A(g)(\mathbf{v}) = C\lambda_i(\mathbf{v}) \sum_{\gamma\in\mathcal{M},\,|\gamma|\geq 1} q_{\gamma,i} \int_{\mathbb{R}^{3|\gamma|}} \lambda^\gamma(\mathbf{w})g^\gamma(\mathbf{w})\,d\mathbf{w},$$

where $q_{\gamma,i} \geq 0$ are constant coefficients, $\gamma \in \mathcal{M}$, $|\gamma| \geq 1$, $1 \leq i \leq N$.

We introduce (5.47) and (5.50) in (5.49). Consequently, for $\mathbf{v} \in \mathbb{R}^3$ a.e.,

$$a(\|\Lambda_B g\|)(\Lambda_B g)_i(\mathbf{v}) - Q_i^-(g)(\mathbf{v}) = [R_i^A(g)(\mathbf{v}) - R_i(g)(\mathbf{v})]g_i(\mathbf{v}) + T_i(g)(\mathbf{v}), \quad (5.51)$$

where

$$T_i(g)(\mathbf{v}) := \lambda_i(\mathbf{v})g_i(\mathbf{v}) \sum_{\gamma\in\mathcal{M},\,|\gamma|\geq 1} (a_0 c_{\gamma,i} - Cq_{\gamma,i}) \int_{\mathbb{R}^{3|\gamma|}} \lambda^\gamma(\mathbf{w})g^\gamma(\mathbf{w})d\mathbf{w}.$$

Now we compare (5.48) and (5.50), by taking advantage of (5.44). It follows that the map $(L^1_{2,+})^N \ni g \mapsto [R_i^A(g) - R_i(g)]g_i \in L^1$ is positive and isotone, $1 \leq i \leq N$. Moreover, because of the form of $T_i(g)$, if $a_0 > 0$ is sufficiently large, then the mapping $(L^1_{2,+})^N \ni g \mapsto T_i(g)(\mathbf{v}) \in L^1$ is positive and isotone for all i. In this case, by virtue of (5.51), the map $(L^1_{2,+})^N \ni g \mapsto a(\|\Lambda_B g\|)\Lambda_B g - Q_B^-(g) \in X$ is also positive and isotone.

In conclusion, the conditions of Theorem 3.1a) are fulfilled (in the case $\Lambda = \Lambda_1$), so that we are in position to state an improved version of a result announced in [6].

Theorem 5.3. *Suppose that in Problem (5.41), $f_{0,i} \in L^1_{4,+}$, $1 \leq i \leq N$. Then Eq. (5.41) has a unique strong solution $f(t) = (f_1, ..., f_N)$ such that $f_i(t) \in L^1_{4,+}$, $t \geq 0$, and $\|f_i(t)\|_{L^1_4}$ is locally bounded on $\mathbb{R}_+$, $1 \leq i \leq N$. In addition, f_i, $(1 + |\mathbf{v}|^2)f_i \in C(\mathbb{R}_+; L^1)$, $1 \leq i \leq N$,*

$$\|\Lambda_B f(t)\| = \|\Lambda_B f_0\| \quad (t \geq 0), \quad (5.52)$$

and there is a non-decreasing function $\rho_B : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$\left\|\Lambda_B^2 f(t)\right\| \leq \exp(\rho_B(\|f_0\|)t) \left\|\Lambda_B^2 f_0\right\| \quad (t \geq 0). \quad (5.53)$$

Observe that Theorem 5.3 does not state the conservation of mass, momentum and energy, but the conservation (in arbitrary units) of the quantity mass+(total) energy. However, the properties of $f(t)$, cf. Theorem 5.3, allow for checking immediately the separate conservation for each of the above quantities.

Theorem 5.3 was first obtained in [6], under a more restrictive formulation than here. The proof of [6] is more complicated than here, because involves an operator approximation step, as mentioned in Section 1, in the case of Arkeryd's scheme [1].

Here it should be noticed that if Problem (5.41) is particularized to the case of the classical Boltzmann equation, then Theorem 5.3 reduces to the main monotonicity result of [1]. Moreover, in that case, using suitable additional Povzner-like estimations, we can re-obtain the general moment estimations of [1], as application of Proposition 3.1b).

Finally we remark that similar analyses as for Theorems 5.2 and 5.3 can be developed for the models considered in [5] and [8].

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References

[1] L. Arkeryd, On the Boltzmann equation I & II, *Arch. Ration. Mech. Anal.* **45**, 1–34 (1972).
[2] E. Wild, On Boltzmann's equation in the kinetic theory of gases, *Proc. Camb. Philos. Soc.* **47**, 602–609 (1951).
[3] A. Ya. Povzner, The Boltzmann equation in the kinetic theory of gases, *Mat. Sb. (N. S.)* **58**(100), 65–86 (1962) (Russian). *Amer. Math. Soc. Translat. Ser. 2* **47** , 193–216 (1962) (English translation).
[4] M. Lachowicz, M. Pulvirenti, A stochastic system of particles modelling the Euler equations, *Arch. Ration. Mech. Anal.* **109**, no. 1, 81–93 (1990).
[5] B. Wiesen, On a phenomenological generalized Boltzmann equation, *J. Math. Phys.* **33**, no. 5, 1786–1798 (1992).
[6] C. P. Grünfeld, Nonlinear Kinetic Models with Chemical Reactions, in N. Bellomo and M. Pulvirenti Eds., **Modeling in Applied Sciences: A Kinetic Theory Approach**, pp. 173–224, Model. Simul. Sci. Eng. Technol., Birkhäuser, Boston (2000).
[7] E. de Angelis, C. P. Grünfeld, The Cauchy problem for the generalized Boltzmann equation with dissipative collisions, *Appl. Math. Lett.* **14**, no. 8, 941–947 (2001)
[8] E. de Angelis and C. P. Grünfeld, Modeling and analytic problems for a generalized Boltzmann equation for a multicomponent reacting gas, *Nonlinear Anal. Real World Appl.* **4**, no. 1, 189–202 (2003).
[9] N. Bellomo and J. Polewczak, The generalized Boltzmann equation solution and exponential trend to equilibrium, *Transport Theory Statist. Phys.* **26**, no. 6, 661–677 (1997).
[10] M. von Smoluchowski, Drei Vorträge über Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteilchen, *Phys. Z.* **17**, 557–571, 585–599 (1916).
[11] H. Müller, Zur allgemeinen Theorie der raschen Koagulation, *Kolloid-Beih.* **27**, 223–250 (1928).
[12] N. Bellomo and M. Pulvirenti Eds., **Modeling in Applied Sciences: A Kinetic Theory Approach**, Model. Simul. Sci. Eng. Technol., Birkhäuser, Boston (2000).
[13] D. Benedetto, E. Caglioti and M. Pulvirenti, Collective Behaviour of One-Dimensional Granular Media, in N. Bellomo and M. Pulvirenti Eds., **Modeling in Applied Sciences: A Kinetic Theory Approach**, pp. 81-110, Model. Simul. Sci. Eng. Technol., Birkhäuser, Boston (2000).
[14] A. V. Bobylev and C. Cerginani, Self-Similar asymptotics for the Boltzmann equation with inelastic and elastic interactions, *J. Satist. Phys.* **110** 1-2, 333–375 (2003).
[15] H. H. Schaefer, **Banach Lattices and Positive Operators**, Springer-Verlag, New York-Heidelberg (1974).
[16] L. V. Kantorovich and G. P. Akilov, **Functional Analysis**, Pergamon Press, Oxford-Elmsford, N.Y. (1982).
[17] E. Hille and R. S. Phillips, **Functional Analysis and Semi-Groups**, American Mathematical Society, Providence (1974).
[18] D. J. Aldous, Deterministic and stochastic models for coalescence (aggregation and coagulation): A review of the mean-field theory for probabilists, *Bernoulli* **5**, no.1, 3–48 (1999).
[19] Z. A. Melzak, A scalar transport equation, *Trans. Amer. Math. Soc.* **85**, 547–560 (1957).
[20] J. B. McLeod, On the scalar transport equation, *Proc. London Math. Soc. (3)* **14** , 445–458 (1964).
[21] V. A. Galkin, The existence and uniqueness of a solution of the coagulation equation, *Differ. Uravn.* ,**13** no. 8, 1460–1470 (1977) (Russian) *Differ. Equ.* **13** no. 8, 1014–1021 (1978) (English Translation).

[22] M. Aizenman and T. A. Bak, Convergence to equilibrium in a system of reacting polymers, *Comm. Math. Phys.* **65**, no. 3, 203–230 (1979).

[23] A. V. Burobin and V. A. Galkin, Solutions of the coagulation equation, *Differ. Uravn.* **17**, no. 4, 669–677 (1981) (Russian), *Differ. Equ.* **17**, No. 4, 456–462 (1981) (English Translation).

[24] V. A. Galkin and P. B. Dubovskii, Solution of the coagulation equation with unbounded kernels, *Differ. Uravn.* **22**, no. 3, 504–509 (1986) (Russian), *Differential Equ.* **22**, no. 3, 373–378 (1986). (English Translation)

[25] N. J. Kokholm, On Smoluchowski's coagulation equation, *J. Phys. A* **21**, no.3, 839–842 (1988).

[26] J. M. Ball and J. Carr, The discrete coagulation-fragmentation equations: existence, uniqueness, and density conservation, *J. Statist. Phys.* **61**, no. 1-2, 203–234 (1990).

[27] O. J. Heilmann, Analytical solutions of Smoluchowski's coagulation equation, *J. Phys. A* **25**, no. 13, 3763–3771 (1992).

[28] P. B. Dubovskii and I. W. Stewart, Existence, uniqueness and mass conservation for the coagulation-fragmentation equation, *Math. Methods Appl. Sci.* **19**, no.7, 571–591 (1996).

[29] I. W. Stewart, A uniqueness theorem for the coagulation-fragmentation equation, *Math. Proc. Cambridge Philos. Soc.* **107** no. 3, 573–578 (1990).

[30] J. M. C. Clark and V. Katsouros, Stably coalescent stochastic froths, *Adv. in Appl. Probab.* **31**, no. 1, 199–219 (1999).

[31] J. R. Norris, Smoluchowski's coagulation equation: Uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent, *Ann. Appl. Probab.* **9**, no. 1, 78–109 (1999).

[32] M. Escobedo, S. Mischler and B. Perthame, Gelation in coagulation and fragmentation models, *Comm. Math. Phys.* **231**, no. 1. 157–188 (2002).

[33] M. Escobedo, Ph. Laurençot, S. Mischler and B. Perthame, Gelation and mass conservation in coagulation-fragmentation models, *J. Differential Equations* **195**, no. 1, 143–174 (2003).

[34] N. Bellomo and M. Esteban, and M. Lachowicz, Nonlinear kinetic equations with dissipative collisions, *Appl. Math. Lett.* **8**, no. 5. 47–52 (1995).

[35] C. P. Grünfeld, On a class of kinetic equations for reacting gas mixtures with multiple collisions, *C. R. Acad. Sci. Paris Sr. I Math.* **316**, no.9, 953–958 (1993).

[36] C. P. Grünfeld and E. Georgescu, On a class of kinetic equations for reacting gas mixtures, *Mat. Fiz. Anal. Geom.* **2**. no. 3-4. 408–435 (1995) (English).

[37] C. P. Grünfeld and D. Marinescu, On the numerical simulation of a class of reactive Boltzmann type equations, *Transport Theory Statist. Phys.* **26**, no. 3, 287–318 (1997).