INSTITUTUL DE MATEMATICA
"SIMION STOILOW"
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

Local homeomorphisms having local ACL^n inverses

by

Mihai Cristea

Preprint nr. 8/2006

BUCURESTI

Local homeomorphisms having local ACL^n inverses

by

Mihai Cristea*

November, 2006

* Faculty of Mathematics and Informatics, University of Bucharest, str. Academiei 14, RO-010014, Bucharest, Romania
E-mail address: mcristea@fmi.unibuc.ro

Local homeomorphisms having local ACL^n inverses

MIHAI CRISTEA
*University of Bucharest, Faculty of Mathematics and Informatics, str. Academiei 14, RO-010014 Bucharest, Romania**
Email: mcristea@fmi.unibuc.ro

We consider local homeomorphisms between domains from $\mathbf{R}^n$ satisfying condition (N) and having local ACL^n inverses. For such mappings we generalize some basic facts from the theory of quasiregular mappings as the modular inequality of Poleckii and estimates of the modulus of spherical rings from [15], [18] and [19], and we use these facts to extend Zoric's theorem and to calculate the radius of injectivity for this class of mappings.

Keywords: Zoric's theorem, eliminability results.

AMS 2000 Classification No: 30C65

1 Introduction

Throughout this paper D will be a domain in $\mathbf{R}^n$ and we consider maps $f : D \to \mathbf{R}^n$, where $D \subset \mathbf{R}^n$ is a domain. Such a map is said to be of finite distortion if:

1) $f \in W^{1,1}_{loc}(D, \mathbf{R}^n)$.

2) The Jacobian determinant is locally integrable.

3) There exists $K : D \to [0, \infty]$ measurable and finite a.e. such that $||f'(x)||^n \leq K(x) \cdot J_f(x)$.

Notice that when $K \in L^\infty(D)$ we obtain the known class of quasiregular mappings and we refer the reader to [22] and [23] for the basic monographs dedicated to this subject. If the distortion map $K \in L^p_{loc}(D)$ for some $p > n-1$ and $f \in W^{1,n}_{loc}(D, \mathbf{R}^n)$, it is shown in [6] that f is open, discrete. In general, for mappings $f : D \to \mathbf{R}^n$ a.e. differentiable with $J_f(x) \neq 0$ a.e., we can define the outer dilatation $K_0(f)$,

*This research is supported by the Romanian Ministry of Education and Research, CEEx Program, Project 2-CEx06-10/25.07.2006

the inner dilatation $K_I(f)$ and the linear dilatation K_f by $K_0(f)(x) = \frac{\|f'(x)\|^n}{|J_f(x)|}$, $K_I(f)(x) = \frac{|J_f(x)|}{l(f'(x))^n}$ and $K_f(x) = \frac{\|f'(x)\|}{l(f'(x))}$ in the points $x \in D$ where f is differentiable with $J_f(x) \neq 0$ and we see that $K_I(f) \leq K_0(f)^{n-1}$ a.e.

If $Q \in BMO(D)$ and $max\{K_0(f)(x), K_I(f)(x)\} \leq Q(x)$ a.e. in D, we say as in [18] that f is a Q-map. In [18] for Q-homeomorphism the following modular estimates are established:

1) $M(f(\Gamma)) \leq M_{K_I(f)}(\Gamma)$.

2) $M_{K_0(f)^{n-1}}(\Delta(\overline{B}(x,r), CB(x,R), B(x,R)\setminus\overline{B}(x,r)) \to 0$ when $r \to 0$ and $R > 0$ is kept fixed and $\overline{B}(x,R) \subset D$.

It must be mentioned that the modular inequalities 1) and 2), although are systematically used, appear in [18], [19], [24], [25], [26] in a nonexplicit form. Also, in [19], [24], [25] are considered non-injective maps for which 1) and 2) holds. However, these maps are ACL^n maps and in the class of local homeomorphisms satisfying condition (N) and having local ACL^n inverses our results are stronger. The methods and the technique of the modular estimates 1) and 2) developed in [24], [25], [26] were also later considered in [15], [9], [13], [14], [4], [5] for mappings of finite distortion and satisfying condition (A). Such maps $f : D \to \mathbf{R}^n$ are maps of finite distortion for which there exists $A : [0,\infty) \to [0,\infty)$ smooth, strictly increasing, with $A(0) = 0$, $\lim_{t\to\infty} A(t) = \infty$, $exp(A \circ K_0(f)) \in L^1_{loc}(D)$, $\int_1^\infty \frac{A'(t)}{t}dt = \infty$ and there exists $t_0 > 0$ such that $A'(t)t$ increases to infinity for $t \geq t_0$. In [13] it is shown that such non-constant maps are open, discrete and in [15] are proved the modular estimates 1) and 2) for this class of mappings. Zoric's type theorems and eliminability results are considered in [9], [14], [4], [5] for such mappings. A local homeomorphism which is a map of finite distortion and satisfying condition (A) also satisfies condition (N) and has local ACL^n inverses (see [8], [15]), hence our extensions will be sharp. We prove the following generalization of a well known theorem of Poleckii from the theory of quaisregular mappings:

Theorem 1. Let $n \geq 2$, $f : D \to \mathbf{R}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses and let Γ be a path family from D. Then $M(f(\Gamma)) \leq M_{K_I(f)}(\Gamma)$.

We also give estimates of the modulus of the spherical ring.

Theorem 2. Let $n \geq 2, x \in \bar{D}, 0 < \delta < a$ such that $B(x,a) \subset D$ if $x \in IntD$, $C_{x,\delta,a,D} = \{z \in D|\ \delta < \|z - x\| < a\}$, $\Gamma_{x,\delta,a,D} = \{\gamma : [0.1] \to C_{x,\delta,a,D}$ path $|\gamma$ joins $S(x,\delta)$ with $S(x,a)$ in $C_{x,\delta,a,D}\}$ and let $w : D \to [0,\infty|$ be measurable and finite a.e. Suppose that one of the following conditions hold:

1) $\int_{B(x,a)\bigcap D} exp(A \circ w)(z)dz < \infty$ for some Orlicz map A.

2) There exists $M > 0$ and $0 \leq \alpha < n-1$ such that $\int_{B(x,r)\bigcap D} w(z)dz \leq M \cdot \mu_n(B(x,r)) \cdot (\ln(\frac{a}{r}))^\alpha$ for $0 < r < a$.

3) $n \geq 3$ and there exists $0 \leq \alpha < n-2, M > 0, Q \in L^1(D \cap B(x,a))$ with $w \leq Q$ such that $\fint_{B(x,r)\cap D} |Q(z) - Q_{B(x,r)\cap D}|dz \leq M \cdot (\ln(\frac{a}{r}))^\alpha$ for every $0 < r < a$ and x is a φ point of D, with $\varphi : (0, \frac{a}{e}) \to (0, \infty)$ decreasing and such that $l_2 = \sum_{k=1}^{\infty} \frac{\varphi(ae^{-k})}{k^{n-\alpha-1}} < \infty$.

4) $n = 2$ and there exists $Q \in L^1(D \cap B(x,a))$ with $w \leq Q$ and $M > 0$ such that $\fint_{B(x,r)\cap D} |Q(z) - Q_{B(x,r)\cap D}|dz \leq M$ for $0 < r < a$ and x is a φ point of D, with $\varphi = c$.

Then $\lim_{\delta \to 0} M_\omega(\Gamma_{x,\delta,a,D}) = 0$.

If condition 2) holds and $l_1 = \sum_{k=1}^{\infty} \frac{1}{k^{n-\alpha}}$, then

$$M_\omega(\Gamma_{x,\delta,a,D}) \leq \frac{M \cdot l_1 \cdot V_n \cdot e^n}{(\ln \ln \frac{ae}{\delta})^n}.$$

If condition 3) holds, $Q_0 = \fint_{B(x,a)\cap D} Q(z)dz$ and $l_3 = \sum_{k=1}^{\infty} \frac{1}{k^n}$, then

$$M_\omega(\Gamma_{x,\delta,a,D}) \leq \frac{M \cdot l_1 \cdot V_n \cdot e^n + \omega_{n-1} \cdot Q_0 \cdot l_3 + \omega_{n-1} \cdot M \cdot l_2}{(\ln \ln \frac{ae}{\delta})^n}.$$

If condition 4) holds, then

$$M_\omega(\Gamma_{x,\delta,a,D}) \leq \frac{\pi \cdot M \cdot l_1 \cdot e^2 + 2\pi \cdot Q_0 \cdot l_3 + 2\pi \cdot M \cdot C \cdot (\ln \ln(\frac{a}{\delta}) - 1)}{(\ln \ln \frac{ae}{\delta})^2}.$$

We extend some results of Dairbekov [6] and Rajala [20] concerning the eliminability of the sets of null capacity of local homeomorphisms for quasiregular mappings, respectively for mappings of finite distortion and satisfying condition (A).

Theorem 3 Let $n \geq 3, B \subset D$ closed in D, $f : D \setminus B \to \overline{\mathbb{R}}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses such that $M_{K_I(f)}(B) = 0$. Then f extends to a homeomorphism around each point $b \in B$. If in addition $M_{K_I(f)}(\infty) = 0$ and there exists $r_0 > 0$ such that $CB(0, r_0) \subset D$, then f extends to a homeomorphism around ∞.

We remark that our condition $M_{K_I(f)}(B) = 0$ is apriori weaker than the condition $M_{K_O(f)^{n-1}}(B) = 0$ imposed in the analogue result of Rajala.

Theorem 4 Let $n \geq 3, E \subset D$ be closed, countable, $f : D \setminus E \to \overline{\mathbb{R}}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses and assume that f satisfies in each point $x \in E$ one of the conditions from Theorem 2. Then f extends to a local homeomorphism $\overline{f} : \overline{D} \to \overline{\mathbb{R}}^n$.

We also extend some results from [1], [3] and [4] proving some Zoric's type theorems with "singularities".

Theorem 5 Let $n \geq 3, K \subset \mathbf{R}^n$ be closed, $B \subset \mathbf{R}^n \setminus K$ be closed in $\mathbf{R}^n \setminus K, f : \mathbf{R}^n \setminus (K \cup B) \to \mathbf{R}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses such that $M_{K_I(f)}(B \cup \{\infty\}) = 0$. Then we can extend f to a homeomorphism around each point $b \in B$ and we also denote by f the extend map. We have:

1) If f is unbounded and $\overline{C(f, K \cup B)}$ is compact, there exists $r_0 > 0$ such that $K \subset B(0, r_0)$ and f is injective on $CB(0, r_0)$.

2) If $\overline{C(f, K \cup B)}$ is compact, $Imf \setminus \overline{C(f, K)} \neq \phi$ and $\mathbf{R}^n \setminus \overline{C(f, K)}$ is connected, then f is injective on $\mathbf{R}^n \setminus K$. If f is continuous on K then $f|\overline{\mathbf{R}}^n \setminus K : \overline{\mathbf{R}}^n \setminus K \to \overline{\mathbf{R}}^n \setminus f(K)$ is a homeomorphism and if in addition f can be extended to an open, discrete map on K, then $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is a homeomorphism.

For $K = \phi$ we obtain the following Zoric type theorem.

Theorem 8 Let $n \geq 3, B \subset \mathbf{R}^n$ be closed, $f : \mathbf{R}^n \setminus B \to \overline{\mathbf{R}}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses such that $M_{K_I(f)}(B \cup \infty) = 0$. Then f extends to a homeomorphism $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$.

We find in Lemma 2 and Lemma 3 some conditions in order to ensure that $M_{K_I(f)}(\infty) = 0$ and using these conditions, we find the following generalization of Zoric's theorem.

Theorem 7 Let $n \geq 3, f : \mathbf{R}^n \to \mathbf{R}^n$ be continuous so that there exists $\rho > 0$ such that $f|CB(0, \rho) : CB(0, \rho) \to f(CB(0, \rho))$ is a local homeomorphism satisfying condition (N) and having local ACL^n inverses on $f(CB(0, \rho))$ and suppose that one of the following conditions are satisfied:

a) there exists $0 \leq \alpha < n - 1$ such that $\lim\limits_{r \to \infty} sup \frac{r^n}{(\ln r)^\alpha} \int\limits_{CB(0,r)} K_I(f)(x) \frac{1}{||x||^{2n}} dx < \infty$.

b) there exists $0 \leq \alpha < n - 1$ such that $\lim\limits_{r \to \infty} sup \fint\limits_{B(0,r)} K_I(f)(x) \frac{dx}{(\ln r)^\alpha} < \infty$.

Then, if f is unbounded, there exists $r_0 > 0$ such that f is injective on $CB(0, r_0)$ and if f is open, discrete on $\mathbf{R}^n$, then $f : \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism.

In [21] it is proved that if $n \geq 3, f : \mathbf{R}^n \to \mathbf{R}^n$ is an ACL^n local homeomorphism, having local ACL^n inverses such that $\lim\limits_{r \to \infty} sup \fint\limits_{B(0,r)} K_f(x)^{n-1} dx < \infty$, then f is a homeomorphism. Since $K_I(f)(x) \leq K_f(x)^{n-1}$ a.e., it results that for such maps the relation $\lim\limits_{r \to \infty} \sup \fint\limits_{B(0,r)} K_I(f)(x) dx < \infty$ holds and hence our Theorem 7 extends Perovic's result, since in our theorem we permit the average $\fint\limits_{B(0,r)} K_I(f)(x) dx < \infty$ to have a logarithmic growth to infinity and we still obtain a Zoric's type result. The preceding theorem also extends our earlier result from [4], established for maps of finite distortion and satisfying condition (A).

We extend now a known theorem from the theory of quasiregular mappings concerning the injectivity radius of a local homeomorphism, generalizing a result established by Martio Rickman and Väisälä for quasiregular mappings in [17] and by Koskela, Onninen and Rajala in [14] for mappings of finite distortion and satisfying condition (A).

Theorem 10 Let $n \geq 3, a > 0, f : B(0,a) \to \mathbf{R}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses such that there exists $0 \leq \alpha < n-1$ so that $M_a = \sup_{0<r<a} \fint_{B(0,a)} K_I(f)(x) \frac{dx}{(\ln \frac{a}{r})^\alpha} < \infty$. Then f is injective on $B(0, \delta_a)$, where $\delta_a > a \cdot e \cdot exp(-exp(C \cdot M_a)^{\frac{1}{n}})$, with C a constant depending only on n.

Theorem 12 Let $n \geq 3, a > 0, f : B(0,a) \to \mathbf{R}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses so that there exists $Q \in L^1_{loc}(B(0,a))$ such that $K_I(f) \leq Q$ a.e. in $B(0,a)$ and let $Q_a = \fint_{B(0,\frac{a}{e})} Q(x)dx$ and $M_a = \sup_{0<r<a} \fint_{B(0,a)} |Q(x) - Q_{B(0,a)}| \frac{dx}{(\ln \frac{a}{r})^\alpha}$, where $0 \leq \alpha < n-2$. Then f is injective on $B(0, \delta_a)$, where $\delta_a > a \cdot e \cdot exp(-exp(C_1 M_a + C_2 Q_a)^{\frac{1}{n}})$, with C_1 and C_2 constants depending only on n.

Our paper may be considered as an attempt of considering mappings of finite distortion under minimal assumptions. One of the natural minimal assumption is to consider mappings $f : D \to \mathbf{R}^n$ a.e. differentiable with $J_f(x) \neq 0$ a.e., since for much maps we can calculate a.e. the dilatations $K_O(f), K_I(f)$, and of course, a map of finite distortion and satisfying condition (A) verifies this minimal assumption.

The class of local homeomorphisms $f : D \to \mathbf{R}^n$ satisfying condition (N) and having local ACL^n inverses is a special class of mappings of finite distortion. Indeed, we see from Proposition 1 such that a map $f : D \to \mathbf{R}^n$ is a.e. differentiable with $J_f(x) \neq 0$ a.e., and since f is an a.e. differentiable local homeomorphism, we use Theorem 24.4, page 84 [27] to see that $J_f \in L^1_{loc}(D)$. Let now $g : V \to U$ be a local inverse of f. Since g is an ACL^n homeomorphism, is a.e. differentiable, and if $A = \{y \in V | g \text{ is differentiable in } y \text{ and } J_g(y) = 0\}$, we use Sard's lemma [3] to see that $\mu_n(g(A)) = 0$, and using the fact that f satisfies condition (N), we obtain that $\mu_n(A) = 0$. It results that g is ACL^n, a.e. differentiable on V and $J_g(y) \neq 0$ a.e. in V, and from Theorem 6.1, page 150, [7], we see that $f \in W^{1,1}_{loc}(U, \mathbf{R}^n)$ hence $f \in W^{1,1}_{loc}(D, \mathbf{R}^n)$.

Working in the class of local homeomorphisms satisfying condition (N) and having local ACL^n inverses, we don't need such a map to belong to $W^{1,n}_{loc}(D, \mathbf{R}^n)$, and we don't impose the map K to be in $L^p(D)$ for every $p > 0$, as is supposed in the case of the mappings of finite distortion and satisfying condition (A), or in the case of Q-homeomorphism, and in these conditions, we still obtain important qualitative results based on the modular inequalities from Theorem 1 and 2. An important subclass is the class of ACL^n local homeomorphisms having local ACL^n inverses, since such maps satisfy condition (N), are a.e. differentiable with $J_f(x) \neq 0$ a.e. and the local inverses also satisfy condition (N) and are a.e. differentiable with non-vanishing jacobian a.e.

In Example 1 we find a homeomorphism $f : D \to D'$ satisfying condition (N)

having ACL^m inverses for every $m \geq 1$, which is not a map of finite distortion and satisfying condition (A) or a Q- homeomorphism, which shows that our theory is consistent. More precisely, $f \notin W^{1,p}_{loc}(D, \mathbf{R}^n)$ for every $p > 1$ and the distortion map $K_0(f) \notin L^1_{loc}(D)$. It results that our extensions are effective and lead to the conclusion that many of the assumptions from the theory of mappings of finite distortion and satisfying condition (A) or from the theory of Q-mappings are redundant in order to extend some basic properties of quasiregular mappings.

In the present paper we considered only local homeomorphisms. We used the modular inequalities 1) and 2) and basically the methods from [4], [5], [10], [11], [14], [15], [18], [19], [21] to improve some results concerning Zoric's theorem, the calculus of the radius of injectivity and some eliminability results. Using the usual technique developed in the above paper, we can consider problems such as boundary extension, distortion estimates, equicontinuity results. We don't attack this problems now, in the particular case of local homeomorphisms, since we shall do this thing in full generality in a more general setting. We give here only some extensions to some known theorems from the theory of quasiconformal mappings from the book of Väisälä [27] which are specifically for homeomorphic mappings. In a forthcoming paper we shall consider the problems mentioned before for open, discrete mappings $f : D \to \mathbf{R}^n$ with $\mu_n(B_f) = 0, \mu_n(f(B_f)) = 0$, giving in this way further extensions to the theory of the so called mappings of finite distortion and satisfying condition (A) from [9], [12], [13], [14], [15], [29] and to the theory of Q-mappings from [10], [11], [18], [19], [24], [25], [26].

2 Preliminaries

If Γ is a path family from $\mathbf{R}^n$, we set $F(\Gamma) = \{\rho : \overline{\mathbf{R}}^n \to [0, \infty]$ Borel maps$| \int_\gamma \rho ds \geq 1$ for every $\gamma \in \Gamma\}$, and if $\omega : D \to [0, \infty]$ is finite and measurable a.e., we put $\tilde{\omega} : \mathbf{R}^n \to [0, \infty], \tilde{\omega}(x) = \omega(x)$ if $x \in D, \tilde{\omega}(x) = 0$ if $x \notin D$ and we set $M_\omega(\Gamma)$ $\inf_{\rho \in F(\Gamma)} \int_{\mathbf{R}^n} \rho^n(x)\tilde{\omega}(x)dx$, the ω modulus of the path family Γ and for $\omega = 1$ we obtain the usual modulus. If Γ_1, Γ_2 are path families, we say that $\Gamma_1 > \Gamma_2$ if every path $\gamma \in \Gamma_1$ has a subpath is Γ_2 and as in the classical case, we prove that if $\Gamma_1 > \Gamma_2$, then $M_\omega(\Gamma_1) \leq M_\omega(\Gamma_2)$ and if $\Gamma_1, \Gamma_2, \ldots, \Gamma_i, \ldots$ are path families, we have $M_\omega(\bigcup_{i=1}^\infty \Gamma_i) \leq \sum_{i=1}^\infty M_\omega(\Gamma_i)$. Also, if $\omega_1 \leq \omega_2$, then $M_{\omega_1}(\Gamma) \leq M_{\omega_2}(\Gamma)$. We define for $E, F \subset \overline{D}, \Delta(E, F, D)$ to be the family of all paths, open or not, which joins E with F in D. If $A \subset \overline{D}$ and $\omega : D \to [0, \infty]$ is measurable and finite a.e., we say that A is of zero ω-modulus (and we write $M_\omega(A) = 0$) if the ω-modulus of all paths in D having some limit point in A is zero, and if $\omega \geq 1$ and $M_\omega(A) = 0$, then cap $A = 0$. If $q : [0, 1) \to \mathbf{R}^n$ is an open path and $x \in \mathbf{R}^n$ we say that x is a limit point of q if there exists $t_p \to 1$ such that $q(t_p) \to x$. If $A \subset D$ is at most countable and $\lim_{r \to 0} M_\omega(\Delta(\overline{B}(x, r) \cap D, CB(x, R) \cap D, D) = 0$ for every $x \in A$, we prove as in the classical case that $M_\omega(A) = 0$ and we give such conditions in Theorem 2.

A domain $A \subset \overline{\mathbf{R}}^n$ is a ring if CA has exactly two-components C_0 and C_1 and we denote $A = R(C_0, C_1)$ and $\Gamma_A \quad \Delta(C_0, C_1, \mathbf{R}^n)$. We say that a domain $D \subset \mathbf{R}^n$ has property P_1 in the point $b \in \partial D$ if $M(\Delta(E, F, D)) = \infty$ for every connected sets E and F from D with $b \in \overline{E} \cap \overline{F}$ and we say that D has property P_2 in b if for every $b_1 \in \partial D, b_1 \neq b$, there exists a continua $F \subset D$ and $\delta > 0$ so that $M(\Delta(E, F, D)) \geq \delta$ for every $E \subset D$ connected with $b, b_1 \in \overline{E}$. If $a, b \in \overline{\mathbf{R}}^n$, we set $q(a, b) = \frac{\|a-b\|}{(1+\|a\|^2)^{\frac{1}{2}}(1+\|b\|^2)^{\frac{1}{2}}}$ if $a, b \in \mathbf{R}^n$ and $q(a, \infty) = \frac{1}{(1+\|a\|^2)^{\frac{1}{2}}}$ if $a \in \mathbf{R}^n$, the chordal distance between a and b. Then q is a metric on $\overline{\mathbf{R}}^n$ and if $A, B \subset \overline{\mathbf{R}}^n$ we set $q(A)$ and $q(A, B)$ the diameter of A, respectively the distance between A and B, considering the chordal metric on $\overline{\mathbf{R}}^n$. If $D_1, \ldots, D_j, \ldots$ is a sequence of domains from $\overline{\mathbf{R}}^n$, we set $\ker D_j = \{y \in \overline{\mathbf{R}}^n |$ there exists $V \in \mathcal{V}(y)$ and $j_y \in \mathbf{N}$ so that $V \subset D_j$ for $j \geq j_y\}$.

Given $r > 0$, we set $\mathcal{H}_n(r) = \inf M(\Gamma_A)$, where the infimum is taken over all rings $A = R(C_0, C_1)$ so that C_0 contains 0 and a point $a \in S(0,1)$ and C_1 contains ∞ and a point $b \in S(0, r)$. Then $\mathcal{H}_n : (0, \infty) \to (0, \infty)$ is decreasing, $\lim_{r \to 0} \mathcal{H}(r) = \infty$, $\lim_{r \to \infty} \mathcal{H}(r) = 0$ and if $A = R(C_0, C_1), a, b \in C_0, c, \infty \in C_1$, then $M(\Gamma_A) \geq \mathcal{H}_n(\frac{\|c-a\|}{\|b-a\|})$ (see Theorem 11.9, page 36, [27]). Given $0 < r < 1$ we denote $\lambda_n(r) = inf M(\Gamma_A)$, where the infimum is taken over all rings $A = R(C_0, C_1)$ with $q(C_0) \geq r, q(C_1) \geq r$ and we know from Theorem 12.5 page 38 that $\lambda_n : (0,1) \to (0, \infty)$ is increasing and $\lim_{r \to 0} \lambda_n(r) = 0$.

If $f \in L^1(A)$ for every $A \subset D$ bounded, we set $f_A = \int_A \frac{f(x)dx}{\mu_n(A)}$ for $A \subset D$ bounded and we write $f_A = ⨍_A f(x)dx$. Here μ_n is the Lebesgue measure from $\mathbf{R}^n$ and we denote by m_p the p-dimensional Hausdorff measure from $\mathbf{R}^n$. We say that $f : D \to \mathbf{R}^n$ is of finite mean oscillation in a point $x \in D$ if $\limsup_{r \to 0} ⨍_{B(x,r)} |f(z) - f_{B(x,r)}| dz < \infty$ and we say that f is of bounded mean oscillation on D (and we write $f \in BMO(D)$) if there exists $M > 0$ so that $⨍_B |f(z) - f_B| dz < M$ for every ball $B \subset D$. We denote by $W^{1,p}_{loc}(D, \mathbf{R}^n)$ the Sobolev space of all functions $f : D \to \mathbf{R}^n$ which are locally in L^p together with their first order weak partial derivatives. Using Proposition 1.2, page 6, [23], we see that $f \in C(D, \mathbf{R}^n)$ is ACL^p for some $p > 1$ if and only if f belongs to the Sobolev space $W^{1,p}_{loc}(D, \mathbf{R}^n)$.

If E, F are Hausdorff spaces and $f : E \to F$ is a map, we say that f is open if f carries open stes into open sets, and we say that f is discrete if $f^{-1}(y)$ is discrete or empty for every $y \in F$. If $p : [0,1] \to F$ is a path and $x \in E$, we say that a path $q : [0,1] \to F$ is a lifting of p from x if $q(0) = x$ and $f \circ q = p$ and we say that $q : [0, a) \to E$ is a maximal lifting of p from x if $q(0) = x, 0 < a \leq 1, f \circ q = p|[0, a)$ and a is maximal with this property. We say that $f : E \to F$ lifts the paths if f lifts every path $p : [0,1] \to F$ from every point $x \in E$ with $f(x) = p(0)$. A local homeomorphism $f : E \to F$ is a covering space if for every $y \in F$ there exists $V \in \mathcal{V}(y)$ and Q_i so that $f^{-1}(V) = \cup_{i \in I} Q_i$ and $f|Q_i : Q_i \to F$ is a homeomorphism for every $i \in I$.

If $f : D \to \mathbf{R}^n$ is a map and $b \in \partial D$ we set $C(f, b) = \{w \in \overline{\mathbf{R}}^n |$ there exists $b_p \in$

$D, b_p \neq b, b_p \to b$ so that $f(b_p) \to w\}$ and for $B \subset \partial D$ we set $C(f, B) = \cup_{b\in B} C(f, b)$. If $Q \subset D$, we set for $b \in \partial D$, $C(f, b, Q) = \{w \in \overline{\mathbf{R}}^n |$ there exists $b_p \in Q, b_p \neq b, b_p \to b$ so that $f(b_p) \to w\}$. If $f : D \to \mathbf{R}^n$ is a map, we say that f satisfies condition (N) if $\mu_n(f(A)) = 0$ for every $A \subset D$ with $\mu_n(A) = 0$.

We can extend in a natural way the definition of a local homeomorphism satisfying condition (N) and having local ACL^n inverses to $\overline{\mathbf{R}}^n$ valued maps as follows: Let $f : D \to \overline{\mathbf{R}}^n$. Then f is a local homeomorphism satisfying condition (N) and having local ACL^n inverses if for each $x \in D$ there exists $U \in \mathcal{V}(x)$ and a Möbius map g so that $g \circ (f(U)) : U \to \mathbf{R}^n$ is a local homeomorphism satisfying condition (N) and having local ACL^n inverses.

If $\alpha : [a, b] \to \mathbf{R}^n$ is a rectifiable path, we denote by $s_\alpha : [a, b] \to [0, l(\alpha)]$ its length function, and we have $\alpha = \alpha^0 \circ s_\alpha$, where α^0 is the normal representation of α (see [27], page 5). If $f : D \to \mathbf{R}^n$ is continuous, open, discrete, $\alpha : [a, b] \to D$ is a path and $\beta = f \circ \alpha$ is rectifiable, we can define α^*, a reparametrization of α by the property $\alpha = \alpha^* \circ s_\beta$, and the definition is correct, due to the discreteness of the map f. We have the relations $\beta^0 = f \circ \alpha^*, (\alpha^*)^0 = \alpha^0$.

If $f : D \to \mathbf{R}^n$ is a map, $A \subset D, y \in \mathbf{R}^n$, we put $N(y, f, A) = Card\, f^{-1}(y) \cap A$ and $N(f, A) = \sup_{y\in\mathbf{R}^n} N(y, f, A)$. If $A \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ we put $||A|| = \sup_{||h||=1} ||A(h)||, l(A) = \inf_{||h||=1} ||A(h)||$ and we denote by V_n the volume of the unit ball from $\mathbf{R}^n$ and by ω_{n-1} the area of the unit sphere from $\mathbf{R}^n$.

If $x \in \overline{D}, a > 0$ and $\varphi : (0, \frac{a}{e}) \to (0, \infty)$ is a map, we say that x is a φ-point of D if $\mu_n(B(x, er) \cap D) \leq \varphi(r) \cdot \mu_n(B(x, r) \cap D)$ for every $0 < r < \frac{a}{e}$. If $\varphi(t) = c$ for $t > 0$ small enough, we say as in [11] that D satisfies a doubling condition in x. If $x \in IntD$, then x is a φ-point of D, with $\varphi = c^n$. We shall need the following lemma from [29]:

Lemma A Let $f : D \to \mathbf{R}^n$ be a local homeomorphism $A, B \subset D$ such that $f|A : A \to f(A), f|B : B \to f(B)$ is a homeomorphism, $A \cap B \neq \phi$ and $f(A) \cap f(B)$ is connected. Then $f|A \cup B : A \cup B \to f(A \cup B)$ is a homeomorphism.

3 Proofs of the results

Proposition 1 Let $f : D \to \mathbf{R}^n$ be a local homeomorphism having local ACL^n inverses. Then f is a.e. differentiable on D and $J_f(x) \neq 0$ a.e. on D.

Proof Let $U \subset D$ be a domain so that $f|U : U \to V$ is a homeomorphism and $h = (f|(U))^{-1} : V \to U$ is ACL^n. Then h satisfies condition (N) and is a.e. differentiable (see [22], page 190) and let $A_h = \{y \in V | h$ is not differentiable in $y\}$ and $Z_h = \{y \in V \setminus A_h | J_h(y) = 0\}$. Then $\mu_n(h(A_h)) = 0$ and we see from [2] that $\mu_n(h(Z_h)) = 0$. Then $Int(A_h \cup Z_h) = \phi$ and let $y_0 \in V \setminus (A_h \cup Z_h)$ and $x_0 = h(y_0)$. Then h is differentiable in y_0 and $J_h(y_0) \neq 0$ and $||f(x) - f(x_0) - h'(y_0)^{-1}(x - x_0)|| = ||h'(y_0)^{-1}||(||x - x_0 - h'(y_0)(f(x) - f(x_0))|| = ||h'(y_0)^{-1}|| \cdot ||h(f(x)) - h(f(x_0)) - h'(y_0)(f(x) - f(x_0)|| \to 0$ if $x \to x_0$. It results that f is differentiable in x_0 and

$f^{'}(x_0) = h^{'}(y_0)^{-1}$, hence $J_f(x_0) \neq 0$. We proved that f is differentiable and $J_f(x) \neq 0$ on $U \setminus h(A_h \cup Z_h)$ and the proof is finished.

Proposition 2 Let $f : D \to \mathbf{R}^n$ be a local homeomorphism having local ACL^n inverses and let Γ be a path family in D and $\tilde{\Gamma} = \{\gamma \in \Gamma | f \circ \gamma$ is locally rectifiable and there exists a closed subpath α of γ such that α^* is not absolutely continuous$\}$. Then $M(f(\tilde{\Gamma}) = 0$.

Proof Let U_i, V_i be open sets such that $f(D) \subset \cup_{i=1}^{\infty} V_i$ and $f|U_i : U_i \to V_i$ is a homeomorphism and $g_i = (f|U_i)^{-1} : V_i \to U_i$ is ACL^n for every $i \in N$. Let $\Gamma^{'} = \{\beta \in f(\Gamma) | \beta$ is rectifiable and for every $\alpha \in \Gamma$ such that $\beta = f \circ \alpha$ and every compact interval $I \subset [0, l(\beta)]$ such that $\beta^0(I) \subset V_i$ for some $i \in N$ it results that $\alpha^*|I = g_i \circ \beta^0|I$ is absolutely continuous on $I\}$. Using a theorem of Fuglede (see [27], Theorem 28.2, page 95), we see that $M(f(\Gamma) \setminus \Gamma^{'}) = 0$ and let $\Gamma_1 = \{\gamma \in \Gamma|$ such that $f \circ \gamma \in \Gamma^{'}\}$. Then $f(\Gamma_1) = \Gamma^{'}$ and let $\alpha \in \Gamma_1$ and $\beta = f \circ \alpha$. We have that $\beta^0 = f \circ \alpha^*, \alpha^* : [0, l(\beta)] \to D$ and using the compactness of $[0, l(\beta)]$ we can cover this interval with intervals $I_1, ..., I_m$ such that $\beta^0(I_i) \subset V_{k_i}$ for some $k_1, , k_m \in \mathbf{N}, i = 1, ..., m$. Since α^* is absolutely continuous on each interval $I_1, ..., I_m$ it results that α^* is absolutely continuous on $[0, l(\beta)]$. We proved that $\tilde{\Gamma} \subset \Gamma \setminus \Gamma_1$, hence $M(f(\tilde{\Gamma})) \leq M(f(\Gamma \setminus \Gamma_1)) = M(f(\Gamma) \setminus \Gamma^{'}) = 0$.

Proof of Theorem 1 Let Γ be a path family in D and $D_0, D_1, ..., D_m, ...,$ domains so that $D_m \nearrow D$. Let $\Gamma_k = \{\alpha \in \Gamma | Im\alpha \subset D_k\}$ and $\tilde{\Gamma}_k = \{\alpha \in \Gamma_k | \alpha$ is a closed path, $f \circ \alpha$ is rectifiable and α^* is absolutely continuous$\}$ for $k \in \mathbf{N}$. We fix $k \in \mathbf{N}$ and let $f_k = f|D_k : D_k \to f(D_k)$. We cover $f(D_k)$ with domains V_i such that $D_k \cap f^{-1}(V_i) = \cup_{j=1}^{j(i)} U_{ji}, f|U_{ji} : U_{ji} \to V_i$ is a homeomorphism such that $g_{ji} = (f|U_{ji})^{-1} : V_i \to U_{ji}$ is an ACL^n map for $j = 1, ..., j(i), i \in \mathbf{N}$. Let $A_{ji} = \{y \in V_i | g_{ji}$ is not differentiable in $y\}$, $Z_{ji} = \{y \in V_i \setminus A_{ji} | J_{g_{ji}}(y) = 0\}$ for $j = 1, ..., j(i), i \in \mathbf{N}$ and let $A = \cup_{i=1}^{\infty} \cup_{j=1}^{j(i)} g_{ji}(A_{ji} \cup Z_{ji})$ and $C = \cup_{i=1}^{\infty} \cup_{j=1}^{j(i)} A_{ji} \cup Z_{ji}$. Using Sard's lemma (see [2]), we see that $\mu_n(A) = 0$ and since f satisfies condition (N) and $C = f(A)$ it results that $\mu_n(C) = 0$. We see that A and C are Borel sets of null measure, $f(A) = C$, f is differentiable on $D_k \setminus A$ and $J_f(x) \neq 0$ and $D_k \setminus A$.

Let now $\rho \in F(\tilde{\Gamma}_k)$ and $\rho^{'} : \mathbf{R}^n \to [0, \infty]$ be defined by $\rho^{'}(y) = \sup_{x \in f_k^{-1}(y)} \frac{\rho(x)}{l(f^{'}(x))}$ if $y \in f(D_k) \setminus C, \rho^{'}(y) = \infty$ on C and $\rho^{'}(y) = 0$ otherwise. Then $\rho^{'}(y) = \sup \rho(g_{ji}(y)) \cdot ||g_{ji}^{'}(y)||, j = 1, ..., j(i)$ for every $y \in V_i \setminus C$ and every $i \in \mathbf{N}$. Since $V_i \setminus C$ are Borel sets for $i \in \mathbf{N}$, we see that $g_{ji}^{'}|V_i \setminus C$ are Borel maps for $i \in \mathbf{N}, j = 1, ..., j(i)$, hence $\rho^{'}$ is a Borel map on $V_i \setminus C$ for every $i \in \mathbf{N}$ and this implies that $\rho^{'}$ is a Borel map.

Let now $\alpha \in \tilde{\Gamma}_k, \beta = f \circ \alpha$ and $A_0 = \{t \in [0, l(\beta)] | \beta^0(t) \in C\}$. If $\mu_1(A_0) > 0$, then

$$1 \leq \infty = \int_{A_0} \infty ds \leq \int_0^{l(\beta)} \rho^{'}(\beta^0(t))dt = \int_{\beta} \rho^{'} ds.$$

Suppose now that $\mu_1(A_0) = 0$. Let $B_0 = \{t \in [0, l(\beta)] | \alpha^*$ or β^0 is not differentiable in t or $||\beta^0(t)^{'}|| \neq 1\}$. Then $\mu_1(B_0) = 0$ and let $C_0 = A_0 \cup B_0$. We fix $t \in [0, l(\beta)] \setminus C_0$. Then $\beta^0(t) \in V_i$ for some $i \in \mathbf{N}$ and since $f_k(\alpha^*(t)) = \beta^0(t)$, there exists $j \in \{1, ..., j(i)\}$ such that $\alpha^*(t) \in U_{ji}$ and since $\beta^0(t) \notin C$, we see that g_{ji} is differentiable in $\beta^0(t)$ and $J_{g_{ji}}(\beta^0(t)) \neq 0$. Differentiating in the equality

$\alpha^*(t) = g_{ji}(\beta^0(t))$, we see that $\alpha^{*'}(t) = g'_{ji}(\beta^0(t))(\beta^{0'}(t)) = g'_{ji}(f(\alpha^*(t))(\beta^{0'}(t)) = [f'(\alpha^*(t))]^{-1}(\beta^{0'}(t))$, hence $1 = \|\beta^{0'}(t)\| = \|f'(\alpha^*(t))(\alpha^{*'}(t))\| \geq l(f'(\alpha^*(t))\|\alpha^{*'}(t)\|$. It results that $l(f'(\alpha^*(t))\|\alpha^{*'}(t)\| \leq 1$ for a.e. $t \in [0, l(\beta)]$.

Since α^* is absolutely continuous on $[0, l(\beta)]$ it results that s_{α^*} is absolutely continuous on $[0, l(\beta)]$. Using a change of variable formulae for absolutely continuous and increasing real mappings, we obtain

$$\int_\beta \rho' ds = \int_0^{l(\beta)} \rho'(\beta^0(t))dt = \int_0^{l(\beta)} \rho'(f(\alpha^*(t))dt \geq \int_0^{l(\beta)} \rho(\alpha^*(t))/l(f'(\alpha^*(t)))dt \geq$$

$$\int_0^{l(\beta)} \rho(\alpha^*(t)) \cdot \|\alpha^{*'}(t)\|dt = \int_0^{l(\beta)} (\rho_0(\alpha^*)^0 s_{\alpha^*})(t) \cdot s'_{\alpha^*}(t)dt =$$

$$\int_0^{l(\beta)} (\rho_0\alpha^0 \circ s_{\alpha^*})(t)s'_{\alpha^*}(t)dt = \int_0^{l(\beta)} \rho(\alpha^0(t))dt = \int_\infty \rho ds \geq 1.$$

We proved in both cases that $\int_\beta \rho' ds \geq 1$, hence $\rho' \in F(\tilde{\Gamma}_k)$.

We take now as in [23], Theorem 8.1, page 49, $\eta_p \nearrow \rho'$ Borel maps so that $0 < \eta_p(y) < \rho'(y)$ if $0 < \rho'(y)$. Let $y \in f(D_k) \setminus C$. Then $y \in V_i$ for some $i \in \mathbf{N}$ and $f_k^{-1}(y)$ is a finite set from D_k and we can find $x \in D_k \setminus A$ such that $y = f(x)$ and $\rho'(f(x)) = \frac{\rho(x)}{l(f'(x))}$. For such a point x we have $\eta_p(f(x)) \leq \rho'(f(x)) = \frac{\rho(x)}{l(f'(x))}$, hence the set $Q_p = \{x \in D_k \setminus A | \eta_p(f(x)) \leq \frac{\rho(x)}{l(f'(x))}\}$ is nonempty for every $p \in \mathbf{N}$. We also remark that $f_k^{-1}(y) \cap Q_p \neq \phi$ for every $y \in f(D_k) \setminus C$. Since f is an a.e. differentiable local homeomorphism, we see from [27], Theorem 24.4, page 84 that $J_f \in L^1_{loc}(D)$ and since f satisfies condition (N), we use the change of variable formulae from [7], Theorem 5.23, page 132 and we have

$$\int_{\mathbf{R}^n} \eta_p^n(y)dy = \int_{f(D_k)\setminus C} \eta_p^n(y)dy \leq \int_{f(D_k)\setminus C} \eta_p^n(y)N(y, f, Q_p)dy =$$

$$\int_{f(Q_p)} \eta_p^n(y)N(y, f, Q_p)dy = \int_{Q_p} \eta_p^n(f(x))|J_f(x)|dx \leq$$

$$\int_{Q_p} \rho^n(x)\frac{|J_f(x)|}{l(f'(x))^n}dx \leq \int_{\mathbf{R}^n} \rho^n(x)K_I(f)(x)dx.$$

It results that $\int_{\mathbf{R}^n} \eta_p^n(y)dy \leq M_{K_I(f)}(\tilde{\Gamma}_k)$ for every $p \in \mathbf{N}$ and letting $p \to \infty$, we obtain that $\int_{\mathbf{R}^n} \rho'^n(y)dy \leq M_{K_I(f)}(\tilde{\Gamma}_k)$, hence we see that $M(f(\tilde{\Gamma}_k)) \leq M_{K_I(f)}(\tilde{\Gamma}_k)$

for every $k \in \mathbf{N}$. Using the generalization of Poleckii's lemma given in Theorem 1, we see that $M(f(\Gamma_k)) = M(f(\tilde{\Gamma}_k)) \le M_{K_I(f)}(\tilde{\Gamma}_k) \le M_{K_I(f)}(\Gamma)$ for every $k \in \mathbf{N}$. Now $\Gamma_k \nearrow \Gamma$, hence $f(\Gamma_k) \nearrow f(\Gamma)$ and using a result of Ziemer [28], we see that $M(f(\Gamma_k)) \nearrow M(f(\Gamma))$. It results that $M(f(\Gamma)) \le M_{K_I(f)}(\Gamma)$ and the theorem is proved.

Proof of Theorem 2 If condition 1) holds, we use Theorem 53, page 24, [15] (see also Lemma 2 from [5]) to see that $\lim_{\delta\to\infty} M_\omega(\Gamma_{x,\delta,a,D}) = 0$.

Suppose that condition 2) holds. We can take $x = 0$ and let $\rho : \overline{\mathbf{R}}^n \to [0,\infty]$ be defined by $\rho(z) = 1/\ln\ln\frac{ae}{\delta}\frac{1}{\|z\|}\ln\frac{ae}{\|z\|}$ for $z \in C_{0,\delta,a,D}$, $\rho(z) = 0$ otherwise. Let $t_k = ae^{-k}$, $B_k = B(0,t_k)$, $A_k = C_{0,t_{k+1},t_k,D}$ for $k \in \mathbf{N}$. Then $A_k \subset B_k$, $\|z\|^{-n} \le \frac{V_n e^n}{\mu_n(B_k)}$ if $z \in A_k$, $k+1 < \ln\frac{ae}{\|z\|}$ for every $z \in B_k$ and $\int_{B_k\cap D} \omega(z)dz \le M\mu_n(B_k)k^\alpha$ for every $k \in \mathbf{N}$. Let $m \in \mathbf{N}$ be such that $t_{m+1} < \delta \le t_m$. Since $\rho \in F(\Gamma_{0,\delta,a,D})$, we have

$$M_\omega(\Gamma_{0,\delta,a,D}) \le \int_{\mathbf{R}^n} \rho^n(z)\omega(z)dz \frac{1}{(\ln\ln\frac{ae}{\delta})^n} \cdot \int_{C_{0,\delta,a,D}} \frac{\omega(z)dz}{\|z\|^n(\ln\frac{ae}{\|z\|})^n} \le$$

$$\frac{1}{(\ln\ln\frac{ae}{\delta})^n}\sum_{k=0}^m \int_{A_k} \frac{\omega(z)dz}{\|z\|^n(\ln\frac{ae}{\|z\|})^n} \le \frac{V_n e^n}{(\ln\ln\frac{ae}{\delta})^n}\sum_{k=0}^m \int_{A_k} \frac{\omega(z)dz}{\mu_n(B_k)(\ln\frac{ae}{\|z\|})^n} \le$$

$$\frac{V_n e^n}{(\ln\ln\frac{ae}{\delta})^n}\sum_{k=0}^m \frac{1}{(k+1)^n} \cdot \int_{B_k\cap D} \frac{\omega(z)dz}{\mu_n(B_k)} \le \frac{MV_n e^n l_1}{(\ln\ln\frac{ae}{\delta})^n}.$$

Suppose now that condition 3) holds. Then $n \ge 3$ and $0 < \alpha < n-2$ and let $Q_k = ⨍_{B_k\cap D} Q(z)dz$ for $k \in \mathbf{N}$. Keeping the notations used before, we have that $⨍_{B_k\cap D} |Q(z) - Q_k|dz \le Mk^\alpha$ for $k \in \mathbf{N}$ and let $S_1 = \sum_{k=0}^m \int_{A_k} \frac{|Q(z)-Q_k|}{\|z\|^n(\ln\frac{ae}{\|z\|})^n}dz$ and $S_2 = \sum_{k=0}^m \int_{A_k} \frac{Q_k}{\|z\|^n(\ln\frac{ae}{\|z\|})^n}dz$. We have

$$S_1 \le V_n e^n \sum_{k=0}^m \frac{1}{\mu_n(B_k)} \int_{A_k} \frac{|Q(z) - Q_k|}{(\ln\frac{ae}{\|z\|})^n}dz \le$$

$$V_n e^n \sum_{k=0}^m \frac{1}{\mu_n(B_k)(k+1)^n} \cdot \int_{B_k\cap D} |Q(z) - Q_k|dz \le MV_n e^n l_1.$$

Also, $|Q_{k+1} - Q_k| = \frac{1}{\mu_n(B_{k+1}\cap D)}|\int_{B_{k+1}\cap D}(Q(z) - Q_k)dz| \le \frac{\varphi(t_{k+1})}{\mu_n(B_k\cap D)}\int_{B_k\cap D}|Q(z) - Q_k|dz \le Mk^\alpha\varphi(t_{k+1})$ for every $k \in \mathbf{N}$, hence $Q_k \le Q_0 + \sum_{l=0}^{k-1}|Q_{l+1} - Q_l| \le Q_0 +$

$M \cdot \sum_{l=1}^{k-1} l^{\alpha}\varphi(t_{l+1}) \leq Q_0 + Mk^{\alpha+1}\varphi(t_k)$ for $k \in \mathbf{N}$. Since $\int_{A_k} \frac{dz}{\|z\|^n(\ln \frac{ae}{\|z\|})^n} \leq \frac{\omega_{n-1}}{(k+1)^n}$ for every $k \in \mathbf{N}$, we see that $S_2 \leq \omega_{n-1} \cdot \sum_{k=0}^{m} \frac{Q_k}{(k+1)^n} \leq \omega_{n-1}Q_0 l_3 + M\omega_{n-1}l_2$. Now

$$M_\omega(\Gamma_{0,\delta,a,D}) \leq \int_{\mathbf{R}^n} \rho^n(z)\omega(z)dz \leq \int_{\mathbf{R}^n} \rho^n(z)Q(z)dz \leq \frac{1}{(\ln\ln \frac{ae}{\delta})^n}.$$

$$\int_{C_{0,\delta,a,D}} \frac{Q(z)dz}{\|z\|^n(\ln \frac{ae}{\|z\|})^n} \leq \frac{S_1+S_2}{(\ln\ln \frac{ae}{\delta})^n} \leq (Ml_1V_ne^n + \omega_{n-1}Q_0l_3 + M\omega_{n-1}l_2)/(\ln\ln \frac{ae}{\delta})^n.$$

Suppose now that condition 4) holds. Then $n = 2, \alpha = 0, \varphi = c$ and we follow the proof from [11]. We have $S_2 \leq 2\pi Q_0 l_3 + 2\pi MC\sum_{k=1}^{m}\frac{1}{k} \leq 2\pi Q_0 l_3 + 2\pi MC(\ln m - 1) \leq 2\pi Q_0 l_3 + 2\pi MC(\ln\ln(\frac{a}{\delta}) - 1)$, hence $M_\omega(\Gamma_{0,\delta,a,D}) \leq [M\pi l_1 e^2 + 2\pi Q_0 l_3 + 2\pi MC(\ln\ln(\frac{a}{\delta}) - 1)]/(\ln\ln \frac{ae}{\delta})^2$.

Remark 1 We see that if $x \in IntD$ and $\limsup_{r\to 0} ⨍_{B(x,r)} \omega(z)dz < \infty$, then, for $a > 0$ fixed and small enough, we have $\lim_{\delta\to 0} M_\omega(\Gamma_{x,\delta,a,D}) = 0$. Also, if $\limsup_{r\to 0} ⨍_{B(x,r)\cap D} |Q(z) - Q_{B(x,r)\cap D}|dz < \infty$ (i.e. Q has finite mean oscillation in x) and the domain D satisfies a doubling condition in $x \in \overline{D}$, it results that $\lim_{\delta\to 0} M_\omega(\Gamma_{x,\delta,a,D}) = 0$, and this case was proved in [11]. Of course, condition 3) is satisfied when $Q \in BMO(D)$, and this case was proved in [18], Lemma 2.2, page 52. Also, for $x \in IntD$, we can take in case 3) $\varphi = e^n$ and we see that $\sum_{k=1}^{\infty} \frac{1}{k^{n-\alpha-1}}$ is convergent for $0 \leq \alpha < n-2$. For $n \geq 4$ and $0 \leq \alpha < n-3$, we can take in case 3) $\varphi(t) = \ln\frac{a}{t}$ for $t \in (0, \frac{a}{e})$ and we see that $l_2 = \sum_{k=1}^{\infty} \frac{\varphi(ae^{-k})}{k^{n-\alpha-1}} \leq \sum_{k=1}^{\infty} \frac{1}{k^{n-\alpha-2}} < \infty$.

Lemma 1 Let D, D' be domains from $\mathbf{R}^n$, $f : D \to D'$ a diffeomorphism, $\omega : D' \to [0,\infty]$ be measurable and finite a.e. and let Γ be a path family from D. Then $M_\omega(f(\Gamma)) \leq M_{\omega\circ f\cdot K_I(f)}(\Gamma)$. If f is conformal, i.e. if $K_I(f)(x) = K_O(f)(x)$ in D, then $M_\omega(f(\Gamma)) = M_{\omega\circ f}(\Gamma)$.

Proof Let $\rho \in F(\Gamma)$ and let $\rho' : \mathbf{R}^n \to [0,\infty]$ be defined by $\rho'(y) = \rho(f^{-1}(y))$ $\|(f^{-1})'(y)\|$ for $y \in D'$, $\rho'(y) = 0$ for $y \notin D'$. Then $\rho' \in F(f(\Gamma))$ and we have

$$M_\omega(f(\Gamma)) \leq \int_{f(D)} \omega(y)\rho'^n(y)dy \int_{f(D)} \omega(y)\rho^n(f^{-1}(y))\|(f^{-1})'(y)\|^n dy =$$

$$\int_{f(D)} \omega(y)\rho^n(f^{-1}(y))\frac{K_I(f)(f^{-1}(y))}{|J_f(f^{-1}(y))|}dy =$$

$$\int_{f(D)} \omega(f(f^{-1}(y))) \cdot \rho^n(f^{-1}(y)) \cdot K_I(f)(f^{-1}(y))|J_{f^{-1}}(y)|dy =$$

$$\int_D \omega(f(x))\rho^n(x)K_I(f)(x)dx.$$

Since $\rho \in F(\Gamma)$ was chosed arbitrarily, we proved that $M_\omega(f(\Gamma)) \leq M_{\omega o f \cdot K_I(f)}(\Gamma)$.

If f is conformal, then also f^{-1} is conformal, hence $K_I(f)(f^{-1}(y)) \cdot K_I(f^{-1})(y) = K_0(f^{-1})(y)K_I(f^{-1})(y) = 1$ for every $y \in D'$. Using the result proved before, we see that $M_\omega(f(\Gamma)) \leq M_{\omega o f \cdot K_I(f)}(\Gamma) \leq M_\omega(f(\Gamma))$.

We proved that $M_\omega(f(\Gamma)) = M_{\omega o f \cdot K_I(f)}(\Gamma)$.

Lemma 2 Let $D \subset \mathbf{R}^n$ be an unbounded domain, $\omega : D \to [0, \infty]$ be measurable and finite a.e. so that there exists $0 \leq \alpha < n-1$ such that $\limsup\limits_{r\to\infty} \frac{r^n}{(\ln r)^\alpha} \int\limits_{CB(0,r)\cap D} \frac{\omega(y)}{||y||^{2n}}dy < \infty$, and let $\Gamma_{rs} = \Delta(\overline{B}(0,r)\cap D, CB(o,s)\cap D, D)$ for $0 < r < s$. Then $\lim\limits_{s\to\infty} M_\omega(\Gamma_{rs}) = 0$ for $1 < r$ fixed.

Proof Let $g : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n, g(x) = \frac{x}{||x||^2}$ for $x \neq 0, g(0) = \infty$. Then g is conformal and let $\Lambda_{rs} = \Delta(g(D) \cap \overline{B}(0, \frac{1}{s}), g(D) \cap CB(0, \frac{1}{r}), g(D))$ for $1 < r < s$. Then $\frac{r^n}{(\ln r)^\alpha} \int\limits_{g(D)\cap B(0,\frac{1}{r})} \omega(g(x))dx = \int\limits_{g(D)\cap B(0,\frac{1}{r})} \omega(g(x)).J_{g-1}(g(x)) \cdot J_g(x)dx \cdot \frac{r^n}{(\ln r)^\alpha} = \frac{r^n}{(\ln r)^\alpha} \int\limits_{g(D)\cap B(0,\frac{1}{r})} \omega(y)J_{g-1}(y)dy = \frac{r^n}{(\ln r)^\alpha} \int\limits_{D\cap CB(o,r)} \frac{\omega(y)}{||y||^{2n}}dy < M$ for $r \geq r_0$.

Using Theorem 2, condition 2) and Lemma 1, we see that $M_\omega(\Gamma_{rs}) = M_{\omega o g}(\Lambda_{rs}) \to 0$ if $r > 1$ is kept fixed and great enough and $s \to \infty$.

Lemma 3 Let $D \subset \mathbf{R}^n$ be an unbounded domain, $\omega : D \to [0, \infty]$ be measurable and finite a.e. so that there exists $0 \leq \alpha < n-1$ such that $\limsup\limits_{r\to\infty} \int\limits_{B(0,r)\cap D} \omega(x)dx/r^n$ $(\ln r)^\alpha < \infty$. Then there exists $r_0 \geq 1$ such that $\lim\limits_{s\to\infty} M_\omega(\Gamma_{rs}) = 0$ for $r \geq r_0$ fixed, where Γ_{rs} is the path family from Lemma 2.

Proof Lat $M > 0$ and $r_0 \geq 2$ be such that $\int\limits_{B(0,r)\cap D} \omega(x)dx < Mr^n(\ln r)^\alpha$ for $r \geq r_0$. We have for $r \geq r_0$:

$$\frac{r^n}{(\ln r)^\alpha} \int\limits_{CB(0,r)\cap D} \frac{\omega(x)}{||x||^{2n}}dx = \frac{r^n}{(\ln r)^\alpha} \sum_{k=0}^{\infty} \int\limits_{D\cap(B(0,r\cdot 2^{k+1})\setminus B(0,r\cdot 2^k))}$$

$$\frac{\omega(x)}{||x||^{2n}}dx \leq \frac{r^n}{(\ln r)^\alpha} \sum_{k=0}^{\infty} \frac{1}{2^{2kn}r^{2n}} \int\limits_{B(0,r2^{k+1})\cap D} \omega(x)dx \leq \frac{M}{r^n(\ln r)^\alpha}.$$

$$\sum_{k=0}^{\infty} \frac{(2^{k+1}r)^n}{2^{2kn}}(\ln(r2^{k+1}))^\alpha \frac{M}{r^n(\ln r)^\alpha} \sum_{k=0}^{\infty} \frac{2^{kn}2^n r^n}{2^{2kn}}.$$

$$\frac{((k+1)\ln 2 + \ln r)^\alpha}{(\ln r)^\alpha} \leq 2^n \sum_{k=0}^{\infty} \frac{M(k+2)^\alpha}{2^{kn}} \leq M \cdot 2^n \cdot 3^\alpha \cdot \sum_{k=0}^{\alpha} \frac{k^\alpha}{2^{kn}} < \infty.$$

We apply now Lemma 2.

Remark 2 The conditions from Lemma 2 and Lemma 3 imply that $M_\omega(\infty)=0$. Also, if $A\subset\overline{D}$ is a countable set and ω satisfies one of the conditions from Theorem 2 in each point from A it results that $M_\omega(A)=0$.

Proposition 3 Let $n\geq 3, B\subset\overline{D}$ such that $B\cap D$ is closed in $D, f: D\setminus B\to\overline{\mathbf{R}^n}$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses such that $M_{K_I(f)}(B)=0$. Let $x\in D\setminus B, p:[0,1)\to\mathbf{R}^n, p(t)=(1-t)f(x)+ty$ for $t\in[0,1)$ be such that $\overline{Imp}\cap\overline{C(f,\partial D\setminus B)}=\phi, q:[0,1)\to D\setminus B$ be a path such that $q(0)=x, f\circ q=p$ and there exists a domain $U\subset D\setminus B$ such that $Imq\subset U, f|U: U\to f(U)$ is a homeomorphism and let g be its inverse. Then, if D is bounded, there exists $z=\lim\limits_{t\to1}q(t)\in D, r_0>0$ and a local inverse of $f, g_{r_0}: f(U)\cup B(y,r_0)\to U\cup g_{r_0}(B(y,r_0))\subset D$ extending g, with $g_{r_0}(y)=z_0$. If D is unbounded and $M_{K_I(f)}(\infty)=0$, there exists $z=\lim\limits_{t\to1}q(t)\in D\cup\{\infty\}, r_0>0$ and a local inverse of $f, g_{r_0}: f(U)\cup B(y,r_0)\to U\cup g_{r_0}(B(y,r_0))$, extending g, with $g_{r_0}(y)=z$ and in the case $z=\infty, g_{r_0}(B(y,r_0))$ is a neighborhood of ∞.

Proof. Step 1. Suppose that D is bounded. Let d be the line containing $Imp, \delta_0=d(y,\overline{C(f,\partial D\setminus B)})>0$ and let $r_0=\min\{\delta_0,\|f(x)-y\|\}$. We see that for $0<r<r_0, S(y,r)$ intersects d in two points, S_r (south pole) and N_r (north pole), and we choose S_r so that $d(f(x),S_r)<d(f(x),N_r)$ for $0<r<r_0$. We cover $S(y,r)\setminus N_r$ with disjoint meridians starting from S_r and we also denote by Δ_r the set of all such meridians for $0<r<r_0$. Let $S_r=p(s_r)$ and we consider the spherical caps $C_{s,r}$ with center y and radius r opening from S_r and having opening angles s for $0<r<r_0$. Let $t(r)$ be the supremum of those opening angles s for which g extends to $C_{s,r}$ as a homeomorphism for $0<r<r_0$. Then $g(C_{t(r),r})\subset D$ and if $t(r)<\pi$, we can find $y_r\in\partial C_{t(r),r}$ and $y_p\in C_{t(r),r}, y_p\to y_r$ such that $g(y_p)\to z\in B$ and if $t(r)=\pi$, we take $y_r=N_r$. We set $E_{r,\epsilon}=\overline{g(B(y_r,\epsilon)\cap C_{t(r),r})}, E_r=\bigcap\limits_{\epsilon>0}E_{r,\epsilon}$ for $0<r<r_0$ and $\epsilon>0$. Then $E_{r,\epsilon}$ are compact subsets from $\overline{D}, E_{r,\epsilon}\cap\partial D\subset B$, $f(E_{r,\epsilon}\cap(D\setminus B))\subset B(y_r,\epsilon)$ hence E_r is a compact subset from $\overline{D}, E_r\cap\partial D\subset B$ and $f(E_r\cap(D\setminus B))=\{y_r\}$ for $0<r<r_0$. Since $capB=0$ and f is a local homeomorphism on $D\setminus B$ it results that $CardE_r=1$ and if $t(r)<\pi, E_r=\{b_r\}$ with $b_r\in B$, for $0<r<r_0$.

Let $A=\{r\in(0,r_0)|E_r\cap B\neq\phi\}, \Gamma'_r=\{\gamma:[0,1)\to\overline{C_{t(r),r}}$ path $|\gamma(0)\in S_r, \gamma(1)=y_r, \gamma([0,1))\subset C_{t(r),r}\}$ for $r\in(0,r_0)$ and let $\Gamma_r=\{\gamma:[0,1)\to D$ path $|\gamma(0)=q(s_r)$ and there exists $\gamma'\in\Gamma'_r$ with $f\circ\gamma=\gamma'$ and γ has at least a limit point in $B\}$ for $r\in(0,r_0)$. Let $\Gamma'=\bigcup\limits_{r\in A}\Gamma'_r, \Gamma=\bigcup\limits_{r\in A}\Gamma_r$. Then $M_{K_I(f)}(\Gamma)=0$ and we see from [27], Theorem 10.2, page 28, that there exists a constant c_n depending only on n such that $c_n\int\limits_A\frac{dr}{r}\leq M(\Gamma')$. Since $\Gamma'>f(\Gamma)$, we have $c_n\int\limits_A\frac{dr}{r}\leq M(\Gamma')\leq M(f(\Gamma))\leq M_{K_I(f)}(\Gamma)=0$, hence $\mu_1(A)=0$.

We proved that for $r\in(0,r_0)\setminus A$, g extends to a homeomorphism on $S(y,r)$ and that $B_r=g(S(y,r))$ bounds a Jordan domain D_r from D. Let $I=p([s_{r_0},1))$. Let

$W = \bigcup_{r\in(0,r_0)\setminus A} \Delta_r \cup I$. Then g extends to a homeomorphism on W and $Q = g(W) \subset D$.

Step 2. Arguing as in [29], [30], we consider for simplicity the case $n = 3$. We can take $p(1) = 0$ and the polar coordinates (ρ, φ, ψ) so that the coordinates of a point from the segment I are $\psi = 0, 0 < \rho < \rho_0$. We consider for $0 < \rho_1 < \rho_0, 0 < \varphi_0 < 2\pi$ the maximal surfaces $L^{'}(\varphi_0) = \{z||0 < \rho_1 < \rho < \rho_0, \varphi = \varphi_0, 0 \leq \psi < \psi(\varphi_0)\}$ for which the components $L(\varphi_0)$ of $f^{-1}(L^{'}(\varphi_0))$ intersecting $q([s_{r_0}, 1))$ are mapped homeomorphically by f on $L^{'}(\varphi_0)$. If $0 < \psi(\varphi_0) < \pi$, we can find at least a point $y_{\varphi_0} \in L^{'}(\varphi_0) \cap \{z|\psi = \psi(\varphi_0), \varphi = \varphi_0\}$ such that if $\Gamma^{'}_{\varphi_0} = \{\gamma : [0, 1] \to L^{'}(\varphi_0)$ path $|\gamma(0) \in I, \gamma(1) = y_{\varphi_0}\}$, it results that every maximal lifting of some paths $\gamma^{'} \in \Gamma^{'}_{\varphi_0}$ starting from some point from $q([s_{r_0}, 1))$ has at least a limit point in B and let Γ_{φ_0} be the family of all such paths. Let $\Lambda = \bigcup_{0<\psi(\varphi)<\pi} \Gamma_\varphi, \Lambda^{'} = \bigcup_{0<\psi(\varphi)<\pi} \Gamma^{'}_\varphi$ and $E = \{\varphi \in [0, 2\pi)|0 \leq \psi(\varphi) < \pi\}$. Then $M_{K_I(f)}(\Lambda) = 0$ and as in [29], [30] we show that there exists a constant a_n (depending in general on n) so that $M(\Gamma^{'}) \geq a_n\mu_1(E)$. We have $a_n\mu_1(E) \leq M_{K_I(f)}(\Lambda) = 0$, hence $\mu_1(E) = 0$. Letting $\rho_1 \to 0$, we can find $E \subset [0, 2\pi)$ with $\mu_1(E) = 0$ so that $L^{'}(\varphi_0) = \{z|0 < \rho < \rho_0, \varphi = \varphi_0, 0 < \psi < \pi\}$ has the property that the component $L(\varphi_0)$ of $f^{-1}(L^{'}(\varphi_0))$ intersecting $q([s_{r_0}, 1))$ is mapped homeomorphically onto $L^{'}(\varphi_0)$ for $\varphi \in [0, 2\pi) \setminus E$. Let $G = \bigcup_{\varphi\in[0,2\pi)\setminus E} L(\varphi)$. Then $G \subset D$ and $H = f(G) = \bigcup_{\varphi\in[0,2\pi)\setminus E} L^{'}(\varphi)$ and let $M = W \cup H$. We see that $f|Q : Q \to W$ and $f|G : G \to H$ is a homeomorphism and from Lemma A we see that $f|Q \cup G : Q \cup G \to M$ is a homeomorphism, hence g extends to a homeomorphism on M and $g(M) = Q \cup G \subset D$.

Step 3. We can suppose that $r_0 \notin A$ hence $B_{r_0} \subset \partial(Q \cup G)$. Let $K = B(y, r_0) \setminus M$. Then K is nowhere disconnecting, hence $C(g, b)$ is a continua from D for every $b \in B(y, r_0)$ and since $capB = 0, f(C(g, b) \setminus B) = \{b\}$ for every $b \in B(y, r_0)$ and f is a local homeomorphism on $D \setminus B$, we see that $C(g, b)$ is a point for every $b \in B(y, r_0)$. We can therefore extend the map g to a continuous map defined on $B(y, r_0)$. Now, since f is a local homeomorphism around each point from $\partial(Q \cup G) \setminus B$ and $f(\partial(Q \cup G) \setminus (B \cup B_{r_0})) \subset K$, it results that also $\partial(Q \cup G) \setminus (B \cup B_{r_0})$ is nowhere disconnecting. Then $\overline{Q \cup G} = \overline{D_{r_0}}$ and $C(f, b, Q \cup G)$ is connected for every $b \in D_{r_0}$.

We define $F : D_{r_0} \to \mathcal{P}(\overline{\mathbf{R}}^n)$ by $F(b) = C(f, b, Q \cup G)$ for $b \in D_{r_0}$. Then $F(x) = f(x)$ on $D_{r_0} \setminus B$ and let us show that F is injective on D_{r_0}, i.e. $F(b_1) \cap F(b_2) = \phi$ if $b_1 \neq b_2$, $b_1, b_2 \in D_{r_0}$. Indeed, otherwise there exists $b_1, b_2 \in D_{r_0}, b_1 \neq b_2$ so that $F(b_1) \cap F(b_2) \neq \phi$ and let $z \in F(b_1) \cap F(b_2)$. We can find $a_p, c_p \in Q \cup G, a_p \to b_1, c_p \to b_2$ such that $f((a_p) \to z, f(c_p) \to z$. Let $\epsilon > 0$ be such that $f(a_p), f(c_p) \in B(z, \epsilon)$ for $p \geq p_\epsilon$. Then $a_p, c_p \in g(B(z, \epsilon) \setminus K)$ for $p \geq p_\epsilon$ and letting $p \to \infty$, we see that $b_1, b_2 \in g(B(z, \epsilon) \setminus K)$. Letting $\epsilon \to 0$, we obtain that $b_1, b_2 \in C(g, z) = g(z)$ hence $b_1 = b_2$. We proved that F is injective on D_{r_0}.

Now $f|D_{r_0} \setminus B : D_{r_0} \setminus B \to B(y, r_0) \setminus F(B)$ is a homeomorphism. Suppose that $capF(B) > 0$ and let $\Gamma_0 = \Delta(B_{r_0}, B, D_{r_0}), \Gamma^{'}_0 = \Delta(S(y, r_0), F(B), B(y, r_0))$. We see that $\Gamma^{'}_0 \subset f(\Gamma_0)$, hence $0 < M(\Gamma^{'}_0) \leq M(f(\Gamma_0)) \leq M_{K_I(f)}(\Gamma_0) = 0$, and we reached a contradiction. It results that $capF(D_{r_0} \cap B) = 0$ and since $F(b)$ is a connected set for every $b \in D_{r_0}$, we obtain that $F(b)$ is a point for every $b \in D_{r_0}$, hence $F : D_{r_0} \to B(y, r_0)$ is a homeomorphism whose inverse is an extension of g.

The proof is similar in the case when D is unbounded.

Remark 3 The proof of Proposition 3 is the key for obtaining Zoric's type theorems and is based on the ideas of Zoric [29], [30]. The arguments used in Step 1 are basically the same as in the papers of Agard and Marden [1], Cristea [3], Dairbekov [6] and Rajala [21]. The arguments from [1] and [3] remain valid only in the case when B is a finite set. In [21] there is a gap in the proof of Theorem 1 from [21] in applying Lemma A, so we had to go back to the original proof of Zoric from [29] and [30] in Step 2.

Proof of Theorem 3 Let $B_0 = \{b \in B | f$ cannot be extend to a homeomorphism around $b\}$ and suppose that $B_0 \neq \phi$ and let $b \in B_0$. Let $x_k \in D \setminus B, x_k \to b$ and $y_k = f(x_k) \to y$, and using if necessary a Möbius transform, we can presume that $y \in \mathbf{R}^n$. Since $capB = 0$, we can take $\rho > 0$ such that $S(b, \rho) \cap B = \phi$ and we can suppose that $x_k \in B(b, \rho)$ for every $k \in \mathbf{N}$ and $r_0 = d(y, f(S(b, \rho)) > 0$. For each $k \in \mathbf{N}$ there exists g_k a local inverse of f around x_k and let $t_k = sup\{s > 0 | g_k$ exists on $B(y_k, s)$ and $Im\ g_k \subset B(b, \rho) \setminus B\}$ for $k \in \mathbf{N}$. We also denote by g_k the extension of g_k on $B(y_k, t_k)$ for every $k \in \mathbf{N}$. If $t_k = \infty$, then $C(g_k, \infty) \subset \overline{B}(b, \rho)$ is a connected set and since $capB = 0$ and there exists $C \subset D \setminus B$ at most countable so that f takes finite values on $D \setminus (B \cup C)$, we see that $CardC(g_k, \infty) = 1$, hence g_k maps $\overline{\mathbf{R}}^n$ homeomorphic onto a proper subset of $\mathbf{R}^n$, which is topologically impossible. It results that $t_k < \infty$ for every $k \in \mathbf{N}$.

Let $D_k = g_k(B(y_k, t_k))$ for $k \in \mathbf{N}$. Suppose that there exists $k \in \mathbf{N}$ so that D_k has a boundary point $b_0 \in B_0 \cap B(b, \rho)$. We can find $b_m \in D_k$ so that $c_m = f(b_m) \to c \in S(y_k, t_k)$. Let $\rho_0 > 0$ be such that $S(b_0, \rho_0) \cap B = \phi$ and let $r_1 = d(c, f(S(b_0, \rho_0))$ and we can suppose that $b_m \in B(b_0, \rho_0)$ for $m \subset \mathbf{N}$ and $r_1 > 0$. Let $p_m : [0, 1] \to B(c, r_1), p_m(t) = (1 - t)c_m + tc$ for $t \in [0, 1]$ and let $q_m : [0, 1) \to B(b_0, \rho_0), q_m = g_k \circ p_m|[0, 1)$ for $m \in \mathbf{N}$. We see that $C(g_k, c)$ is a connected set from $\overline{B}(b_0, \rho_0)$ and since f is a local homeomorphism on $\overline{B}(b_0, \rho_0) \setminus B, capB = 0$ and $f(C(g_k, c) \setminus B) = \{c\}$, it results that $CardC(g_k, c) = 1$. Since $b_m \in D_k, f(b_m) = c_m \in B(y_k, t_k)$ for every $m \in \mathbf{N}$ and $c_m \to c$ we se that $C(g_k, c) = \{b_0\}$, hence $\lim_{t \to 1} q_m(t) = b_0$. Using Proposition 3, we can extend g_k to a homeomorphism around b_0 and this contradicts the fact that $b_0 \in B_0$. It results that each domain D_k has at least a boundary point $\alpha_k \in S(b, \rho)$ for $k \in \mathbf{N}$. Let y_k be so that $||y_k - y|| < \frac{r_0}{4}$. Then $B(y, \frac{r_0}{2}) \subset B(y_k, t_k)$. Indeed, if $\alpha \in B(y, \frac{r_0}{2})$, then $||y_k - \alpha|| \leq ||y_k - y|| + ||y - \alpha|| < \frac{r_0}{4} + \frac{r_0}{2} = r_0 - \frac{r_0}{4} \leq ||f(\alpha_k) - y|| - ||y_k - y|| \leq ||f(\alpha_k) - y_k|| = t_k$, hence $\alpha \in B(y_k, t_k)$. We can therefore suppose that $y_k \in B(y, \frac{r_0}{4})$ and that $B(y, \frac{r_0}{2}) \subset B(y_k, t_k)$ for every $k \in \mathbf{N}$. It results that g_k exists on $B(y, \frac{r_0}{2})$ for every $k \in \mathbf{N}$ and let $\rho_k = \sup\{s > 0 | g_k$ extends on $B(y, s)$ and $Im\ g_k \subset B(b, \rho) \setminus B\}$ and we denote this extension of g_k on $B(y, \rho_k)$ by h_k for $k \in \mathbf{N}$. As before, we show that $\rho_k < \infty$ and each domain $Q_k = h_k(B(y, \rho_k))$ has at least a boundary point $a_k \in S(b, \rho)$ for $k \in \mathbf{N}$.

We have $\rho_k = ||f(a_k) - y|| \leq r_0$ for $k \in \mathbf{N}$ and taking if necessary a subsequence, we can suppose that there exists $a \in S(b, \rho)$ such that $a_k \to a$, and let $O_a \in V_a, V_a \in$

$\mathcal{V}(f(a))$ be such that $f|O_a : O_a \to V_a$ is a homeomorphism, $V_a = B(f(a), \beta)$ for some $\beta > 0$ and that $a_k \in O_a$ for $k \in \mathbf{N}$. Since $B(y, \frac{r_0}{2}) \subset B(y_k, t_k)$ and $B(y_k, t_k)$ is the domain of definition of the map g_k, we see that h_k and g_k are defined on $B(y, \frac{r_0}{2})$ and $g_k|B(y, \frac{r_0}{2}) = h_k|B(y, \frac{r_0}{2})$. Then $x_k = g_k(y_k) = h_k(y_k) \in h_k(B(y, \frac{r_0}{2})) \subset Q_k$ for $k \in \mathbf{N}$.

Let $Q = \bigcup_{k=1}^{\infty} Q_k$. We show that f is injective on Q. Indeed, if this is false, we can find $k, m \in \mathbf{N}, k \neq m$ and $a \in Q_k, c \in Q_m$ such that $f(a) = f(c)$. If $Q_k \cap Q_m \neq \phi$, we see from Lemma A that f is injective on $Q_k \cup Q_m$ and we reached a contradiction, since $f(a) = f(c)$. If $Q_k \cap Q_m \neq \phi$, we use the fact that $a_k \in \partial Q_k \cap Q_a$ to see that $Q_k \cap Q_a \neq \phi$ and since $f(Q_k) \cap f(Q_a)$ is connected and nonempty, we see from Lemma A that f is injective on $Q_k \cup Q_a$. Now $(Q_k \cup Q_a) \cap Q_m \neq \phi$ and $f(Q_k \cup Q_a) \cap f(Q_m) = B(f(a), \beta) \cup B(y, \rho_k) \cap B(y, \rho_m)$ is connected and nonempty, and applying again Lemma A, we obtain that f is injective on $Q_k \cup Q_a \cup Q_m$. We reached again a contradiction, and hence f is injective on Q. Then $f(Q) = B(y, \lambda)$, with $\lambda = \sup_{k \in \mathbf{N}} \rho_k$ and let $h : B(y, \lambda) \to Q$ be its inverse. Then $b = \lim_{k \to \infty} x_k = \lim_{k \to \infty} h_k(y_k) = \lim_{k \to \infty} h(y_k) = h(y)$, and using Brouwer's theorem, we see that $Q \in \mathcal{V}(b)$ and $f|Q : Q \to B(y, \lambda)$ is a homeomorphism. We reached a contradiction, since we supposed that $b \in B_0$. We therefore proved that we can extend f to a homeomorphism around each point $b \in B$.

Suppose now that $b = \infty$. Then there exists $r_0 > 0$ so that $CB(0, r_0) \subset D$ and let $\Gamma = \{\gamma : [0, 1) \to CB(0, r_0)$ path $|\gamma$ has some limit point in $B \cup \{\infty\}\}$. Let $g : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n, g(x) = \frac{x}{||x||^2}$ if $x \in \mathbf{R}^n \setminus \{0\}, g(0) = \infty, g(\infty) = 0$. Then g is conformal and $g(CB(0, r_0)) = B(0, \frac{1}{r_0}) \setminus \{0\}$. We also see that if $A, B \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$, then $l(A \circ B) = \inf_{||h||=1} ||A(B(h))|| \geq \inf_{||h||=1} l(A)||B(h)|| = l(A) \cdot l(B)$.

We have:

$$K_I(f \circ g)(x) = \frac{|J_{f \circ g}(x)|}{l((f \circ g)'(x))^n} = \frac{|J_f(g(x)) \cdot J_g(x)|}{l(f'(g(x))(g'(x)))^n} \leq$$

$$\frac{|J_f(g(x))|}{l(f'(g(x)))^n} \cdot \frac{|J_g(x)|}{l(g'(x))^n} = K_I(f \circ g)(x)$$

for a.e. $x \in B(0, \frac{1}{r_0})$, hence $K_I(f \circ g) \leq K_I(f) \circ g$ on $B(0, \frac{1}{r_0})$. Using Lemma 1, we see that $M_{K_I(f \circ g)}(g(\Gamma)) \leq M_{K_I(f) \circ g}(g(\Gamma)) \quad M_{K_I(f)}(\Gamma) \quad 0$. From what we have proved before, it results that $f \circ g$ extends to a homeomorphism around 0, hence f extends to a homeomorphism around ∞.

Remark 4 The preceding theorem holds if f satisfies condition (N) and has local ACL^n inverses only in some neighborhood of each point $b \in B \bigcup \{\infty\}$.

Proposition 4 Let $n \geq 3, K \subset \mathbf{R}^n$ be closed, $B \subset \mathbf{R}^n \setminus K$ be closed in $\mathbf{R}^n \setminus K, f : \mathbf{R}^n \setminus (K \cup B) \to \overline{\mathbf{R}}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses such that $M_{K_I(f)}(B) = 0$. Then we can extend f to a homeomorphism around each point $b \in B$ and we also denote by f the extended map. If also $M_{K_I(f)}(\infty) = 0$, extending if necessary the map f to a homeomorphism around ∞ and also denoting by f the extended map, we can lift every path $p : [0,1] \to \mathbf{R}^n \setminus \overline{C(f,K)}$ from every x with $f(x) = p(0)$.

Proof From Theorem 3, we can extend f to a homeomorphism around each $b \in B$. We can easily see that if f can be extended to a homeomorphism around ∞, then we can lift every path $p : [0,1] \to \mathbf{R}^n \setminus \overline{C(f,K)}$ from every x with $f(x) = p(0)$.

Suppose now that we cannot extend f to a homeomorphism around ∞. Let $x \in \mathbf{R}^n \setminus K$ be so that $y = f(x) \in \mathbf{R}^n \setminus \overline{C(f,K)}, r_0 = d(y, \overline{C(f,K)})$ and let $\overline{y} \in B(y, r_0)$. Let $p : [0,1] \to B(y, r_0)$ be defined by $p(t) = (1-t)y + t\overline{y}$ for $t \in [0,1]$ and suppose that we cannot lift p from x. Then there exists $0 < a < 1$ and $q : [0,a) \to \mathbf{R}^n \setminus K$ a path so that $q(0) = x, f \circ q = p|[0,a)$ and ∞ is a limit point of q. We can find Q open so that $Im\ q \subset Q$ and $f|Q : Q \to f(Q)$ is a homeomorphism and let $g : f(Q) \to Q$ be its inverse. Using Proposition 3 applied to the domain $D = \mathbf{R}^n \setminus K$, we can find $r > 0$ and a local inverse of f, $g_r : f(Q) \cup B(p(a), r) \to Q \cup D_r$ extending g such that $g_r(p(a)) = \infty$ and D_r is the exterior of a Jordan domain and $\infty \in D_r$. It results that we can extend f to a homeomorphism around ∞ and we reached a contradiction. We proved that we can lift p from x, hence we can lift every line $p : [0,1] \to \mathbf{R}^n \setminus \overline{C(f,K)}, p(t) = (1-t)y + t\overline{y}, t \in [0,1], \overline{y} \in B(y, r_0)$ from every point x with $f(x) = p(0)$.

Let now $p : [0,1] \to \mathbf{R}^n \setminus \overline{C(f,K)}$ be a path and let $x \in \mathbf{R}^n \setminus K$ be so that $f(x) = p(0)$ and let V be the component of $\mathbf{R}^n \setminus \overline{C(f,K)}$ containing Imp and let U be the component of $f^{-1}(V)$ containing x. Then $f|U : U \to V$ is a covering space, hence $f|U : U \to V$ lifts the paths and hence f lifts p from x.

Proof of Theorem 5 Suppose that condition 1) holds and let $\rho > 0$ be so that $\overline{C(f, K \cup B)} \subset B(0,\rho)$. Since f is unbounded, we can find $x \in \mathbf{R}^n \setminus K$, such that $f(x) \in CB(0,\rho)$. Let $V = CB(0,\rho)$ and U be the component of $f^{-1}(V)$ containing x. Using Proposition 4, we see that $f|U : U \to V$ lifts the paths, V is simply connected and since f is a local homeomorphism on U, we see that $f|U : U \to V$ is a homeomorphism and let $g : V \to U$ be its inverse. Then $\partial g(V)$ has two components, one being $g(S(o,\rho))$, which bounds a Jordan domain D_ρ, and the other one is $C(g,\infty)$. Since $V \cap C(f, K \cup B) = \phi$, we see that $C(g,\infty)$ is a compact, connected subset of $\overline{\mathbf{R}}^n \setminus (K \cup B)$ and since f is a local homeomorphism on $\mathbf{R}^n \setminus (K \cup B)$, it results that $Card C(g,\infty) = 1$ and $C(g,\infty) = \infty$. This implies that $U = C\overline{D}_\rho$ and if $r_0 > 0$ is such that $\overline{D}_\rho \subset B(0, r_0)$, then $CB(0,r_0) \subset U$ and f is injective on $CB(0,r_0)$, $K \subset \overline{D}_\rho \subset B(0,r_0)$ and $\lim_{z \to \infty} f(z) = \infty$.

Suppose that condition 2) holds. We see from Proposition 4 that, extending if necessary f to a homeomorphism around ∞, we can lift every path $p : [0,1] \to \mathbf{R}^n \setminus \overline{C(f,K)}$ from every $x \in \mathbf{R}^n$ with $f(x) = p(0)$. The hypothesis implies that $\mathbf{R}^n \setminus \overline{C(f,K)} \subset f(\mathbf{R}^n \setminus K)$ and hence f is unbounded. We show first that f is injective on $\mathbf{R}^n \setminus E$, where $E = K \cup f^{-1}(\overline{C(f,K)})$. Let $x_1, x_2 \in \mathbf{R}^n \setminus E$ be so that $f(x_1) = f(x_2)$. Keeping the notations used before, we see that $f|U : U \to V$ is a homeomorphism and since $U = C\overline{D}_\rho$ with D_ρ a Jordan domain, it results that U is the single component of $f^{-1}(V)$ and $U = f^{-1}(V)$. Let $y \in V$ and $p : [0,1] \to \mathbf{R}^n \setminus \overline{C(f,K)}$ a path such that $p(0) = f(x_1), p(1) = y$. Let $q_k : [0,1] \to \mathbf{R}^n \setminus K$ be paths such that $q_k(0) = x_k$, $f \circ q_k = p, k = 1,2$. Then $q_k(1) \in f^{-1}(V) = U, f(q_k(1)) = y, k = 1,2$, hence $q_1(1) = q_2(1)$. We use now the property of the uniqueness of path liftings for local homeomorphisms to see that $x_1 = x_2$. We proved that f is injective on $\mathbf{R}^n \setminus E$.

We show that if $a \in \mathbf{R}^n \setminus K$ is so that f is open in a, then $f^{-1}(f(a)) = \{a\}$. Indeed, suppose that there exists $b \in \mathbf{R}^n \setminus K, b \neq a$ such that $f(a) = f(b)$. Let $U_a \in \mathcal{V}(a), U_b \in \mathcal{V}_b$ be disjoint such that $f|U_a : U_a \to f(U_a)$ is a homeomorphism and $f(U_b) \subset f(U_a)$. Since $Int\overline{C(f,K)} = \phi$, we can find $w \in f(U_b) \setminus \overline{C(f,K)}$, hence we can find $\alpha \in U_a \setminus E, \beta \in U_b \setminus E$ such that $f(\alpha) = f(\beta)$ and we reached a contradiction, since we proved that f is injective on $\mathbf{R}^n \setminus E$. It results that $f^{-1}(f(a)) = \{a\}$ if $a \in \mathbf{R}^n \setminus K$ and since $f(\infty) = \infty$, we see that f is injective on $\overline{\mathbf{R}}^n \setminus K$.

Suppose now that f is continuous on K. Then $f^{-1}(f(K)) = K$ and $f(\mathbf{R}^n \setminus K) = \mathbf{R}^n \setminus f(K)$ and since $f(\infty) = \infty$, we see that $f|\overline{\mathbf{R}}^n \setminus K : \overline{\mathbf{R}}^n \setminus K \to \overline{\mathbf{R}}^n \setminus f(K)$ is a homeomorphism. If in addition f is also open, discrete on K, then $K \subset D_\rho$, and $f|CD_\rho : CD_\rho \to f(CD_\rho)$ is a homeomorphism, hence f is injective on ∂D_ρ and open, discrete on D_ρ and using the univalence on the border theorem from [3], we see that f is injective on $\overline{D}_\rho$ and hence $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is a homeomorphism.

We obtain the following generalization of Zoric's theorem:

Theorem 6 Let $n \geq 3, K \subset \mathbf{R}^n$ be compact, $f : \mathbf{R}^n \to \mathbf{R}^n$ be continuous on $\mathbf{R}^n$ and a local homeomorphism and satisfying condition (N) on $\mathbf{R}^n \setminus K$, having local ACL^n inverses on $f(\mathbf{R}^n \setminus K)$ so that $M_{K_I(f)}(\infty) = 0$. Then, if f is unbounded, there exists $r_0 > 0$ such that f is injective on $CB(0, r_0)$ and if f is open, discrete on K, then $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is a homeomorphism.

Proof Using Theorem 5, condition 1), we only have to show that f is unbounded if f is open, discrete. Let $x \in \mathbf{R}^n \setminus K, U_x \in \mathcal{V}(x), V_x = B(f(x), r)$ such that $f(U_x) = V_x, f(\partial U_x) = \partial V_x$. If $y \in S(f(x), r)$, we let $\gamma_y : [1, \infty) \to \mathbf{R}^n, \gamma_y(t) = (1-t)f(x) + ty$ for $t \geq 1$ and let $E = \{y \in S(f(x), r) | \gamma_y$ cannot be lifted from some point $\alpha \in \partial U_x$ with $y = f(\alpha)\}$. Let $\Gamma' = \{\gamma_y | y \in E\}$ and let Γ be the family of all maximal liftings of some paths from Γ' starting from some point $\alpha \in \partial U_x$ and having ∞ as a limit point. We have $M(\Gamma') \leq M(f(\Gamma)) \leq M_{K_I(f)}(\Gamma) = 0$, hence $M(\Gamma') = 0$ and $\mu_{n-1}(E) = 0$. It results that we can lift a.e. path γ_y from some point $\alpha \in \partial U_x$ and this implies that f is unbounded. Now, there exists $r_0 > 0$ so that $f|CB(0, r_0) : CB(0, r_0) \to f(CB(0, r_0))$ is a homeomorphism. Since f is injective on $S(0, r_0)$ and open, discrete on $B(0, r_0)$, we use the univalence on the border theorem from [3] to see that f is injective on $\overline{B}(0, r_0)$ and hence that $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is a homeomorphism.

Remark 4 Let $f : \mathbf{R}^n \to \mathbf{R}^n, f(x) = x$ if $x \in \overline{B}(0,1), f(x) = \frac{x}{||x||^2}$ if $||x|| > 1$. Then f is bounded, conformal on $\mathbf{R}^n \setminus S(0,1)$, is not open on $S(0,1)$ and it is not injective. This shows that the openness of the map f on the "singular" set K, or the unboundedness of the map f are necessary conditions in Theorem 6.

Proof of Theorem 7 We apply Theorem 6, Lemma 2 and Lemma 3.

Proof of Theorem 8 We see from Theorem 3 that f extends to a local homeomorphism on $\overline{\mathbf{R}}^n$. If U is a component of $f^{-1}(\mathbf{R}^n)$, then $f|U : U \to \mathbf{R}^n$ lifts the paths and hence it is a homeomorphism and if g is its inverse, we see that $g(\mathbf{R}^n)$ has a single boundary component, namely $C(g, \infty)$ and $C(g, \infty)$ is connected. Since f is a local homeomorphism on $\mathbf{R}^n \setminus B$ and $capB = 0$, we see that $C(g, \infty) = \{b\}$ for some $b \in B \cup \{\infty\}$ and hence $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is a homeomorphism.

We also have

Theorem 9 Let $n \geq 3, K \subset \mathbf{R}^n$ be closed, $B \subset \mathbf{R}^n \setminus K$ be closed in $\mathbf{R}^n \setminus K, f : \mathbf{R}^n \setminus (K \bigcup B) \to \overline{\mathbf{R}}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses such that $M_{K_I(f)}(B \bigcup \{\infty\}) = 0$ and suppose that $Int C(f, K) = \phi$, $\overline{C(f, K)}$ is compact and $\mathbf{R}^n \setminus \overline{C(f, K)}$ is connected. Then we can extend f to a homeomorphism around each point $b \in B$ and we also denote by f the extended map.

Suppose that one of the following conditions holds:

1) K is compact.

2) $CardB = j < \infty$.

3) $C(f, \infty)$ is bounded.

Then there exists $m \in \mathbf{N}$ so that $N(f, \mathbf{R}^n \setminus K) \leq m$ and each value $y \in \mathbf{R}^n \setminus \overline{C(f, K)}$ is taken by f by exactly q-times, with $q \leq m$.

Proof We see from Theorem 3 that we can extend f to a homeomorphism around each point $b \in B$, and let $\rho > 0$ be so that $\overline{C(f, K)} \subset B(0, \rho)$ and let $V = CB(0, \rho)$. We use Proposition 4 and the preceedings arguments to see that $f^{-1}(V) \neq \phi$ and let $(U_i)_{i \in \mathbf{N}}$ be the components of $f^{-1}(V)$. Then $f|U_i : U_i \to V$ is a homeomorphism and let $g_i : V \to U_i$ their inverses for $i \in \mathbf{N}$. We see that ∂U_i has two components, one being $g_i(S(0, \rho))$ and the other is $C(g_i, \infty)$, which contains just one point $b_i \in B \cup \{\infty\}$ so that $f(b_i) = \infty, i \in \mathbf{N}$.

If K is compact, we see from Theorem 3 that f can be extended to a homeomorphism around ∞, hence the set $E = (b_i)_{i \in \mathbf{N}}$ cannot have ∞ as a limit point and cannot have some limit point in K, since $\overline{C(f, K)}$ is compact. It results that E can have some limit point $b \in \mathbf{R}^n \setminus K$, and we reached a contradiction, since f is a local homeomorphism in b. We find that the set E is finite. Also, if $C(f, \infty)$ is bounded, the set E cannot have some limit point in $K \cup \{\infty\}$, and we obtain again that E is finite. Let $q = CardE$, and we see that $q \leq j$ if $CardB = j < \infty$.

If we have points $x_1, \ldots, x_m \in \mathbf{R}^n \setminus K$ such that $f(x_1) = f(x_p), p = 1, \ldots, m$, let $Q_i \in V(x_i)$ be disjoint such that $f(Q_i) = W = B(f(x_1), r), i = 1, \ldots, m$ and let $w \in W \setminus \overline{C(f, K)}$. Then we can find $a_i \in Q_i \setminus f^{-1}(f(\overline{C(f, K)}))$ such that $f(a_i) = w, i = 1, \ldots, m$ and let $y \in V$ and $p : [0, 1] \to \mathbf{R}^n \setminus \overline{C(f, K)}$ be a path so that $p(0) = f(a_1), p(1) = y$. We can find $q_i : [0, 1] \to \mathbf{R}^n \setminus K$ paths such that $q_i(0) = a_i, f \circ q_i = p$, $i = 1, \ldots, m$. Since $q_i(1) \in f^{-1}(V) = \bigcup_{i=1}^{q} U_i$ and using the property of the uniqueness of path lifting for local homeomorphism, we find that $m \leq q$.

Proof of Theorem 10 We sketch the proof from [14] and for the sake of completeness we give a slightly elaborate proof. We can take $f(0) = 0$ and we denote by $\mathbf{U}(0, f, r)$ the component of $f^{-1}(B(0, r))$ containing 0 for $r > 0$ and let $r_0 = sup\{r > 0 | \overline{U}(0, f, r) \subset B(0, a)\}$. Let $l_r^* = inf\{z \in \mathbf{R}^n | z \in \partial U(o, f, r)\}$ and $L_r^* = sup\{z \in \mathbf{R}^n z \in \partial U(0, f, r)\}$ for $0 < r < r_0$. Then f maps $\overline{U}(0, f, r)$ homeomorphically, hence f is injective on $\overline{B}(0, l_r^*)$ for $0 < r < r_0$ and $L_r^* \to a$ if $r \to r_0$.

We take $x_r, y_r \in \partial U(0, f, r)$ such that $||x_r|| = L_r^*$ and $||y_r|| = l_r^*$. Then $f(x_r)$ and $f(y_r) \in S(0, r)$ and from Lemma 3.1 [14], there exists $p_r \in B(0, r)$ so that for every $t \in (\frac{r}{2}, \frac{r\sqrt{3}}{2})$ we have $f(x_r) \in B(p_r, t)$ and either 0 or $f(y_r)$ belongs to $B(p_r, t)$, but not both. Since $f(B(0, l_r^*))$ is connected, we can find $z_t \in S(p_r, t) \cap f(B(0, l_r^*))$ and let z_t^* be the unique point from $f^{-1}(z_t) \cap \overline{B}(0, l_r^*)$ for $t \in (\frac{r}{2}, \frac{r\sqrt{3}}{2})$. We denote by $C_t(\phi) \subset S(p_r, t)$ the spherical cap of center z_t and opening angle ϕ. Let $C_t \subset S(p_r, t)$ be the spherical cap of center z_t and opening angle ϕ_t, where ϕ_t is the supremum of all ϕ for which z_t^* component of $f^{-1}(C_t(\phi))$ is mapped homeomorphically onto $C_t(\phi)$, and let C_t^* be the component of $f^{-1}(C_t)$ for $t \in (\frac{r}{2}, \frac{r\sqrt{3}}{2})$. As in [14], we see that $C_t^* \cap S(0, L_r^*) \neq \phi$ and let $x_t^* \in C_t^* \cap S(0, L_r^*)$ and $x_t = f(x_t^*)$ for $t \in (\frac{r}{2}, \frac{r\sqrt{3}}{2})$. Let Γ_t' be the family of all paths joining x_t with z_t in C_t and let $f_t = f|C_t^*$ for $t \in (\frac{r}{2}, \frac{r\sqrt{3}}{2})$. Let $\Gamma' = \bigcup_{t \in (\frac{r}{2}, \frac{r\sqrt{3}}{2})} \Gamma_t', \Gamma = \bigcup_{t \in (\frac{r}{2}, \frac{r\sqrt{3}}{2})} \{f_t^{-1} \circ \gamma | \gamma \in \Gamma_t'\}$.

Let $\rho \in F(\Gamma')$. We see from [27], Theorem 10.2, page 28, that there exists a constant C depending only on n such that $\frac{C}{t} \leq \int_{S(p_r, t)} \rho^n(x) d\sigma$ for $t \in (\frac{r}{2}, \frac{r\sqrt{3}}{2})$.

Integrating over $t \in (\frac{r}{2}, \frac{r\sqrt{3}}{2})$, we find a constant C_1 depending only on n such that $0 < C_1 < M(\Gamma')$. Using Theorem 1 and Theorem 2, condition 2), we find a constant C_2 depending only on n such that $C_1 \leq M(\Gamma') \leq M_{K_I(f)}(\Gamma) \leq \frac{C_2 M_a}{(\ln \ln(\frac{L_r^* e}{l_r^*}))^n}$. Taking

$C_3 = \frac{C_2}{C_1}$, we obtain that $l_r^* \geq L_r^* \cdot e \cdot \exp(-\exp(C_3^{\frac{1}{n}} M_a^{\frac{1}{n}}))$ and letting $r \to r_0$, the theorem is proved.

Theorem 11 Let $n \geq 3$, $f : \mathbf{R}^n \to \mathbf{R}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses, and let $0 \leq \alpha < n-1$ and $M_a = \sup_{0<r<a} \int_{B(0,r)} \frac{K_I(f)(x) dx}{(\ln \frac{a}{r})^\alpha}$ for $a > 0$. Suppose that there exists $r_j \to \infty$ such that $M_{r_j} < \infty$ for $j \in \mathbf{N}$ and suppose that $\lim_{j \to \infty} \inf r_j \cdot e \cdot \exp(-\exp(C' \cdot M_{r_j})^{\frac{1}{n}}) = \infty$, where C is the constant from Theorem 10. Then f is injective on $\mathbf{R}^n$.

Theorem 13 Let $n \geq 3$, $f : \mathbf{R}^n \to \mathbf{R}^n$ be a local homeomorphism satisfying condition (N) and having local ACL^n inverses so that there exists $Q \in L^1_{loc}(\mathbf{R}^n)$ with $K_I(f) \leq Q$ a.e. on $\mathbf{R}^n$ and let $Q_a = \int_{B(0, \frac{a}{e})} Q(x) dx$ and $M_a = \sup_{0<r<a} \int_{B(0,a)} \frac{|Q(x) - Q_{B(0,a)}| dx}{(\ln \frac{a}{r})^\alpha}$ for $a > 0$, where $0 \leq \alpha < n-2$. Suppose that there exists $r_j \to \infty$ such that $M_{r_j} < \infty$ for $j \in \mathbf{N}$ and suppose that $\lim_{j \to \infty} \inf r_j \cdot e \cdot \exp(-\exp(C_1 M_{r_j} + C_2 Q_{r_j})^{\frac{1}{n}}) = \infty$, where C_1 and C_2 are the constants from Theorem 12. Then f is injective on $\mathbf{R}^n$.

Example 1 We take $I = [0, 1]$ and we define the Cantor set $E \subset I$ as follows: At Step 1 we remove $E_1 = (\frac{1}{3}, \frac{2}{3})$, at Step 2 we remove $E_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$. At Step m we remove from each interval J from $I \setminus \bigcup_{k=1}^{m-1} E_k$ an interval having the same center as J and of length $\frac{l(J)}{3}$, and we denote by E_m the union of all such removed interval at Step m. We see that $\mu_1(E_m) = \frac{1}{3}(\frac{2}{3})^{m-1}$ for $m \geq 1$. We continue this infinite process, we let $E = I \setminus \bigcup_{m=1}^{\infty} E_m$ and we see that $\mu_1(E) = 0$.

We take $0 < a < 1$ and we define $\rho : I \to \mathbf{R}$ by $\rho(t) = a^{m-1}$ for $t \in E_m, m \geq 1$. Then ρ is measurable, $\int_0^1 \rho^k(t)dt = \frac{1}{3}\sum_{m=0}^{\infty}(\frac{2}{3})^m a^{mk} < \infty$ for $k \in \mathbf{N}$ and let $g : [0,1] \to \mathbf{R}$ be defined by $g(x) = \int_0^x \rho(t)dt$ for $x \in I$. Then $g \in L^1(I), g' = \rho$ a.e., g is strictly increasing, satisfies condition (N) and $\int_0^1 [g']^k(t)dt < \infty$ for $k \in \mathbf{N}$. If (α, β) is an interval from E_m, then $(g(\alpha), g(\beta))$ is the corresponding interval of length $(\beta-\alpha)a^{m-1}$ for every $m \in \mathbf{N}$, hence $g(E)$ is also a Cantor set with $\mu_1(g(E)) = 0$, obtained in the same way as the Cantor set E, where the union of intervals E_m of total length $\frac{1}{3}(\frac{2}{3})^{m-1}$ is replaced by the union of intervals F_m of total length $\frac{1}{3}(\frac{2a}{3})^{m-1}$ for $m \in \mathbf{N}$. Let $J = g(I)$.

Then $J \subset I, l(J) = \frac{1}{(3-2a)}$ and let $h : J \to I, h = g^{-1}$. We see that if (α, β) is an interval from E_m and $(g(\alpha), g(\beta))$ is the corresponding interval from F_m, then h is differentiable on $(g(\alpha), g(\beta))$ and $h'(y) = \frac{1}{g'(h(y))} = a^{1-m}$ for $y \in (g(\alpha), g(\beta))$. It results that $\int_J h'(t)dt = \frac{1}{3}\sum_{m=0}^{\infty}(\frac{2}{3})^m < \infty$ and if $p > 1$ is fixed, we have $\int_J [h'(t)]^p dt = \frac{1}{3}\sum_{m=0}^{\infty}(\frac{2}{3}a^{1-p})^m = \infty$ if we take $0 < a < (\frac{2}{3})^{\frac{1}{p-1}}$ and we also see that $h' \notin L^p(J_0)$ if $J_0 \subset J$ is an interval with $J_0' \cap g(F) \neq \phi$.

We take now $n \geq 3$, we fix $p > 1$ and we take $0 < a < (\frac{2}{3})^{\frac{1}{p-1}}$. Let $Q = J \times I^{n-1}$ and $f : Q \to I^n$ be defined by $f(x_1, ..., x_n) = (h(x_1), x_2, ..., x_n)$ for $x = (x_1, ..., x_n) \in Q$. Then f is an a.c. differentiable homeomorphism, $J_f(x) \neq 0$ a.e. on Q, f satisfies condition (N) and $f \in W^{1,1}_{loc}(Q, \mathbf{R}^n) \setminus W^{1,p}_{loc}(Q, \mathbf{R}^n)$. The inverse $f^{-1} : I^n \to Q$ is given by $f^{-1}(y_1, ..., y_n) = (g(y_1), y_2, ..., y_n)$ for $y = (y_1, ..., y_n) \in I^n$ hence $f^{-1} \in W^{1,m}_{loc}(I^n, \mathbf{R}^m)$ for $m \in \mathbf{N}$. Since $K_0(f)(x) = h'(x_1)^{n-1}$ a.e. in Q, we see that $K_0(f)$ is not integrable on every interval $J_0 \times I^{n-1} \subset Q$ so that $J_0' \cap g(E) \neq \phi$, hence $K_0(f) \notin L^1_{loc}(Q)$.

4 Homeomorphisms satisfying condition (N) and having ACL^n inverse

The maps involved in the following theorems are considered apriori homeomorphisms, and for such maps we solve problems like eliminability, boundary extension, equicontinuity, modulus of continuity and characterize the limit map. As we said before, the general setting of the theory will be possed in a forthcoming paper for open, discrete maps. The following 10 theorems are the corresponding versions of Theorem 17.3, 17.13, 17.15, 18.1, 18.2, 19.2, 19.4, 21.1, 21.9, 21.10, 21.11,21.13,21.14 from

[27] given for quasiconformal mappings. We give the proof only for a few of them, since the proofs follow the classical line from [27], and we just replace the classical modular inequalities with our improved versions 1) and 2). Similar results given for Q-homeomorphisms can be found in [10], [18], [19], [24], [25], [26] and for mappings of finite distortion and satisfying condition $(\mathcal{A})$ you can see [5], [15].

Theorem 14 Let $D' \subset \overline{\mathbf{R}}^n$ be a domain, $b \in \partial D$ be an isolated point of ∂D, $f : D \to D'$ be a homeomorphism satisfying condition (N) and having an ACL^n inverse so that $M_{K_I(f)}(\{b\}) = 0$. Then f extends to a homeomorphism $\overline{f} : D \cup \{b\} \to D' \cup \{b'\}$.

Proof. Let $r > 0$ be such that $\overline{B}(b, r) \subset D, A = R(\{b\}, CB(b, r)), A' = R(C(f, b), Cf(B(b, r)))$. Then $M_{K_I(f)}(\Gamma_A) = 0$ and $M(\Gamma_{A'}) = M(f(\Gamma_A)) \leq M_{K_I(f)}(\Gamma_A) = 0$ and this implies that $Card C(f, b) = 1$.

Proposition 5 Let C_0, C_1 be continua in $\mathbf{R}^n, A = R(C_0, C_1)$ and $\omega \in L^1_{loc}(\mathbf{R}^n)$. Then $M_\omega(\Gamma_A) < \infty$.

Proof Let $r = d(C_0, C_1)$ and $D \subset \mathbf{R}^n$ so that $\overline{C}_0 \cup \overline{C}_1 \subset D$ and D is compact and let $\rho : \overline{\mathbf{R}}^n \to [0, \infty), \rho(x) = \frac{1}{r}$ if $x \in D, \rho(x) = 0$ if $x \notin D$. Then $\rho \in F(\Gamma_A)$ and $M_\omega(\Gamma_A) \leq \int\limits_{\mathbf{R}^n} \rho^n(x)\omega(x)dx = \frac{1}{r^n} \int\limits_D \omega(x)dx < \infty$.

Theorem 15 Let $D' \subset \overline{\mathbf{R}}^n$ be a domain, $f : D \to D'$ be an ACL^n homeomorphism so that f^{-1} satisfies condition (N), $b \in \partial D$ such that D has property P_1 in b and $\int\limits_{D'} K_I(f^{-1})(y)dy < \infty$. Then $C(f, b)$ has at most one point at which D' is finitely connected.

Proof Suppose that D' is finitely connected at two distinct points b_1', b_2' from $C(f, b)$. Let $x_j \to b, y_j \to b$ so that $f(x_j) \to b_1', f(y_j) \to b_2'$ and let $U_k \in \mathcal{V}(b_k)'$ be balls so that $\overline{U}_1 \cap \overline{U}_2 = \phi, k = 1, 2$. Extracting if necessary a subsequence, we can find $E_1' \subset U_1, F_1' \subset U_2$ connected such that $f(x_j) \in E_1', f(y_j) \in F_1'$ for every $j \in \mathbf{N}$, and let $E_1 = f^{-1}(E_1'), F_1 = f^{-1}(F_1')$. Then $b \in \overline{E}_1 \cap \overline{F}_1$ and let $\Gamma = \Delta(E_1, F_1, D), \Gamma' = \Delta(E_1', F_1', D')$ and $A = R(\overline{U}_1, \overline{U}_2)$. Using Proposition 5, Theorem 1, the P_1 property of the domain D in the point b and the fact that $\Gamma' > \Gamma_A$, we have that $\infty = M(\Gamma) = M(f^{-1}(\Gamma')) \leq M_{K_I(f^{-1})}(\Gamma') \leq M_{K_I(f^{-1})}(\Gamma_A) < \infty$, and we reached a contradiction.

Theorem 16 Let $D' \subset \overline{\mathbf{R}}^n$ be a domain, $f : D \to D'$ a homeomorphism satisfying condition (N) and having ACL^n inverse, $b \in \partial D$ so that D is locally connected in b, $K_I(f)$ satisfies one of the conditions from Theorem 2 in b and suppose that D' has property P_2 in some point from $C(f, b)$. Then f has a limit in b.

Proof Suppose that $C(f, b)$ contains two distinct point b_1', b_2' and that D' has property P_2 in b_1'. Let $F \subset D'$ be compact and $\delta > 0$ be so that $M(\Delta(E, F, D')) \geq \delta$ for every $E \subset D'$ connected so that $b_1', b_2' \in \overline{E}$. Since D is locally connected in b, there exists $U_j \in \mathcal{V}(b)$ such that $U_j \cap D$ is connected, $U_j \subset B(b, r_j), j \in \mathbf{N}$ and $r_j \to 0$. Let $\Gamma_j = \Delta(U_j \cap D, f^{-1}(F), D), \Gamma_j' = \Delta(f(U_j \cap D), F, D')$ for $j \in \mathbf{N}$. Using Theorem 1 and 2 and the P_2 property of D' in b_1', we have $\delta \leq M(\Gamma_j') \leq M_{K_I(f)}(\Gamma_j) \to 0$, and we reached a contradiction.

Theorem 17 Let $\omega : D \to [0,\infty]$ be measurable and finite a.e. and satisfying in each point $x \in D$ one of the conditions from Theorem 3, W be a family of homeomorphisms $f : D \to D_f \subset \overline{\mathbf{R}}^n$ satisfying condition (N) and having ACL^n inverses so that $K_I(f) \leq \omega$ for $f \in W$ and there exist $r > 0$ so that each $f \in W$ omits at least two points a_f, b_f with $q(a_f, b_f) \geq r$. Then W is equicontinuous.

Proof Let $x_0 \in D$ and $0 < \epsilon < r$. Let $Q_0 = B(x_0, \alpha), Q_1 = B(x_0, \beta), 0 < \alpha < \beta$, be so that $\overline{B}(x_0, \beta) \subset D$ and let $A = R(\overline{Q}_0, CQ_1)$. Then $f(A) = R(f(\overline{Q}_0), Cf(Q_1))$ and $q(Cf(Q_1)) \geq q(a_f, b_f) \geq r, q(f(\overline{Q}_0)) \geq q(f(x), f(x_0))$ for every $x \in \overline{Q}_0$ and every $f \in W$. Let $f \in W, x \in \overline{Q}_0$ and $t = min\{r, q(f(x), f(x_0))\}$. We keep $\beta > 0$ fixed and we choose $\alpha > 0$ small enough so that $M_\omega(\Gamma_A) \leq \lambda_n(\epsilon)$. Then $\lambda_n(t) \leq M(\Gamma_{f(A)}) = M(f(\Gamma_A)) \leq M_{K_I(f)}(\Gamma_A) \leq M_\omega(\Gamma_A) \leq \lambda_n(\epsilon)$, and since λ_n is increasing, we see that $t \leq \epsilon$. Since $\epsilon < r$, we obtain that $t = q(f(x), f(x_0))$, hence $q(f(x), f(x_0)) \leq \epsilon$ for every $x \in \overline{U}_0 = \overline{B}(x_0, \alpha)$ and every $f \in W$, i.e. the family W is equicontinuous in x_0.

Theorem 18 Let $\omega : D \to [0,\infty]$ be measurable and finite a.e. and satisfying in each point $x \in D$ one of the conditions from Theorem 2, W be a family of homeomorphisms $f : D \to D_f \subset \overline{\mathbf{R}}^n$, satisfying condition (N) and having ACL^n inverses so that $K_I(f) \leq \omega$ for every $f \in W$, and suppose that one of the following conditions hold:

1) there exists $x_1, x_2 \in D$ and $r > 0$ so that each $f \in W$ omits a point a_f with $q(a_f, f(x_i)) \geq r, i = 1, 2$.

2) there exists $x_i \in D$ and $r > 0, i = 1, 2, 3$ so that $q(f(x_i), f(x_j)) \geq r$ for $i \neq j, i, j = 1, 2, 3$ and every $f \in W$.

Then W is equicontinuous.

Theorem 19 Let $\omega \in L^1_{loc}(D)$ satisfying in each point $x \in D$ one of the conditions from Theorem 2, $f_j : D \to D_j \subset \overline{\mathbf{R}}^n$ be homeomorphisms satisfying condition (N) and having ACL^n inverses so that $K_I(f_j) \leq \omega$ for every $j \in \mathbf{N}$ and suppose that $f_j \to f$. Then, if $CardImf \geq 3$, it results that $f : D \to D'$ is a homeomorphism onto a domain $D' \subset \overline{\mathbf{R}}^n$. If $f_j \to f$ uniformly on the compact subsets from D, then f is either a homeomorphism onto a domain $D' \subset \overline{\mathbf{R}}^n$, or a constant c. If we suppose in addition that f_j are ACL^n maps for every $j \in \mathbf{N}$, that $Card\partial D \geq 2$ and there exists $\omega_0 \in L^1_{loc}(\mathbf{R}^n)$ satisfying in each point one of the conditions from Theorem 2 so that $K_I(f_j^{-1}) \leq \omega_0$ for every $j \in \mathbf{N}$, then D' is a component of $KerD_j$ if $f : D \to D'$ is a homeomorphism, and if $f = c$, then $c \in C(KerD_j \cup KerCD_j)$. Also, if $f : D \to D'$ is a homeomorphism and $F \subset D'$ is compact, there exists $j_0 \in \mathbf{N}$ such that $F \subset D_j$ for $j \geq j_0$ and $f_j^{-1} \to f^{-1}$ uniformly on F.

Theorem 20 Let D, D' be domains in $\mathbf{R}^n$ with $Card\partial D \geq 2, \omega \in L^1_{loc}(D), \omega_0 \in L^1_{loc}(D')$ satisfying in each point one of the conditions from Theorem 2, let $f_j : D \to D'$ be ACL^n homeomorphisms with ACL^n inverses so that $K_I(f_j) \leq \omega, K_I(f_j^{-1}) \leq \omega_0$ for every $j \in \mathbf{N}$ and $f_j \to f$. Then the convergence is uniformly on the compact subsets from D and the limit map is either a homeomorphism onto D', or a constant $c \in \partial D'$. In the first case, $f_j^{-1} \to f^{-1}$ uniformly on the compact subsets from D', and the second case can occur only if ∂D has only one component, or infinitely many components or exactly two pointwise components.

Theorem 21 Let D, D' be domains in $\mathbf{R}^n$ with $Card \partial D' \geq 2, F \subset D$ be compact and let W be a family of homeomorphisms $f : D \to D'$ satisfying condition (N) and having ACL^n inverses so that there exists $\omega \in L^1_{loc}(D)$ satisfying in each point one of the conditions from Theorem 2 such that $K_I(f) \leq \omega$ for every $f \in W$. Then, for every $\epsilon > 0$, there exists $\delta > 0$ so that $q(f(F)) \leq \epsilon$ if $q(f(F), \partial D') \leq \delta$. If $Card \partial D \geq 3$ and ∂D has exactly k components, with $2 \leq k \leq \infty$, $Card F \geq 2, f_j$ are ACL^n maps for every $j \in \mathbf{N}$ and there exists $\omega_0 \in L^1_{loc}(D')$ satisfying in each point one of the conditions from Theorem 2 so that $K_I(f_j^{-1}) \leq \omega_0$ for every $j \in \mathbf{N}$, then there exists $\delta > 0$ such that $q(f(F)) \geq \delta$ and $q(f(F), \partial D') \geq \delta$ for every $f \in W$.

Theorem 22 Let D, D' be domains in $\mathbf{R}^n, x \in D, a = d(x, \partial D)$, W be a family of homeomorphisms $f : D \to D'$ satisfying condition (N) and having ACL^n inverses so that there exists $\omega \in L^1_{loc}(D)$ satisfying one of the conditions 2), 3) or 4) from Theorem 2 in the point x so that $\omega \in L^1(B)$ for every ball $B \subset D$ and $K_I(f) \leq \omega$ for every $f \in W$. Then there exists a continuous, increasing function $\theta_{n,x,\omega} : (0,1) \to (0, \infty)$ such that $\lim_{r \to 0} \theta_{n,x,\omega}(r) = 0$, $\lim_{r \to 1} \theta_{n,x,\omega}(r) = \infty$ and $\|f(y) - f(x)\| / d(f(x), \partial D') \leq \theta_{n,x,\omega}(\frac{\|y-x\|}{a})$ for every $y \in B(x,a)$ and every $f \in W$. If $\sup_{B \subset D ball B} \int \omega(z) dz < \infty$, then the function $\theta_{n,x,\omega}$ does not depend on x.

Proof Let $y \in B(x, a), d' = d(f(x), \partial D'), A = R(\overline{B}(x, \|y-x\|, CB(x,a))$ and $f \in W$. Then $A \subset D, f(A) = R(C_0, C_1)$, where $C_0 = f(\overline{B}(x, \|y-x\|)), C_1 = Cf(B(x,a))$ and $f(x), f(y) \in C_0, C_1$ contains ∞ and a point $b' \in \partial D'$ so that $d' = \|f(x) - b'\|$. Using Theorem 1 and 2 and Theorem 11.9 page 36, [27], we have $\mathcal{H}_n(d'/\|f(y) - f(x)\|) \leq M(\Gamma_{f(A)}) \leq M_{K_I(f)}(\Gamma_A) \leq M_\omega(\Gamma_A)$. If ω satisfies condition 2) or 3) from Theorem 2 in x, we take $\theta_{n,x,\omega}(r) = 1/\mathcal{H}_n^{-1}(C/(\ln \ln \frac{e}{r})^n)$ for $r > 0$ and if ω satisfies condition 4) from Theorem 2 in x, we take $\theta_{n,x,\omega}(r) = 1/\mathcal{H}_n^{-1}(C_1/(\ln \ln \frac{e}{r})^2 + C_2/(\ln \ln \frac{e}{r}))$ for $r > 0$, where the constants C, C_1, C_2 depend on x, n, ω. In the case $\sup_{B \subset D ball B} \int \omega(z) dz = M < \infty$, we take $C = M l_1 V_n e^n$, which is a constant depending only on n and ω, hence $\theta_{n,\omega}(r) = 1/\mathcal{H}_n^{-1}(C/(\ln \ln \frac{e}{r})^n)$ depends only on n and ω.

Theorem 23 Let W be a family of homeomorphisms $f : B(0,1) \to B(0,1)$ satisfying condition (N), having ACL^n inverses, so that $f(0) = 0$ for every $f \in W$ and there exists $\omega \in L^1(B(0,1))$ satisfying one of the conditions 2), 3) or 4) from Theorem 2 for $x = 0$ and $a = 1$ and so that $K_I(f) \leq \omega$ for every $f \in W$. Then there exists $\varphi_{n,\omega} : (0,1) \to (0,1)$ continuous, increasing, with $\lim_{r \to 0} \varphi_{n,\omega}(r) = 0$, $\lim_{r \to 1} \varphi_{n,\omega}(r) = 1$ such that $\|f(x)\| \leq \varphi_{n,\omega}(x)$ for every $x \in B(0,1)$ and every $f \in W$.

REFERENCES

1) S. Agard and A. Marden, A removable singularity theorem for local homeomorphisms, Indiana Univ. Math. Journal, 28, 5(1970),455-461.

2) M. Cristea, A generalization of Sard's theorem. A Jacobian condition for interiority, Demonstratio Mathematica, 21(1988),399-405.

3) M. Cristea, A generalization of a theorem of Zoric, Bull. Math. Soc. Sci. Roumaine, 34,3(1990), 207-217.

4) M. Cristea, Mappings of finite distortion: Zoric's theorem, equicontinuity results, Preprint 17, dec. 2004, Instit. Matem. Acad. Române.

5) M. Cristea, Mappings of finite distortion: Boundary extension, Preprint 18, dec. 2004, Instit. Matem. Academiei Române.

6) N. S. Dayrbekov, Removable singularities for locally quasiconformal mappings, Siberian Math. Journal, 33,1(1992),159-161.

7) I. Fonseca and W. Gangbo, Degree Theory in Analysis and Appl.,, Oxford Univ. Press, 1995.

8) J. Heinonen and P. Koskela, Sobolev mappings with integrable dilatation, Arch. Rat. Mech. Anal, 125(1993), 81-97.

9) I. Holopainen and P. Pankka, Mappings of finite dilatation: Global homeomorphism theorem, Ann. Acad. Sci. Fenn., Math., 29(2004),135-151.

10) A. Ignatiev and V. Ryazanov, On the boundary behavior of space mappings, Preprint 350, 2003, Reports Math. Dept. Univ. of Helsinki.

11) A. Ignatiev and V. Ryazanov, Finite mean oscillation in the mapping theory, Preprint 332, 2002, Reports Math. Dept. Univ. of Helsinki.

12) T. Iwaniec and G. Martin, Geometric function theory and nonlinear analysis, Oxford Math. Monogr., 2001.

13) J. Kauhanen, P. Koskela, J. Malý, J. Onninen and X. Zhong, Mappings of finite distortion: Sharp Orlicz conditions, J. Eur. Math. Soc. (JEMS),5(2003),95-105.

14) P. Koskela, J. Onninen and K. Rajala, Mappings of finite distortion: Injectivity radius of local homeomorphisms, Future Trends in Geometric Function Theory, RNC Workshop Jyväskylä 2003, Rep. Univ. Jyväskylä, Dep. Math. Stat., 92(2003),169-174.

15) P. Koskela and J. Onninen, Mappings of finite distortion: Capacity and modulus inequalities, Preprint 257, Univ. of Jyväskylä, 2002.

16) J. Manfredi and E. Villamor, An extension of Reshetnyak's theorem, Indiana Univ. Math. Journal, 43,3(1998),49-69.

17) O. Martio, S. Rickman and J. Väisälä, Definitions for quasiregular mappings, Ann. Acad. Sci. Fenn., Ser AI, Math., 448 (1969),1-40.

18) O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, On Q-homeomorphisms, Ann. Acad. Sci. Fenn., Math., 30(2005),49-69.

19) O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, Mappings of finite lenght distortion, Preprint 322, 2002, Reports Math. Dept. Univ. of Helsinki.

20) M. Perovic, On the problem of radius of injectivity for the mappings quasiconformal in the mean, Glasnik Math., 20(40)(1985),345-348.

21) K. Rajala, Mappings of finite distortion: Removable singularities for locally homeomorphic mappings, Proc. Amer. Math. Soc., 123(2004),3251-3258.

22) Y.G. Reshetnyak, Space Mappings of Bounded Distortion, Transf. of Math. Monogr., Amer. Math. Soc., Providence, RI, 73(1989).

23) S. Rickman, Quasiregular mappings, Ergebnisse der Mathematik und ihrer Grenzgebiete,26, Springer-Verlag,1993.

24) V. Ryazanov, U. Srebro and E. Yakubov,To the theory of BMO-quasiregular mappings, Dokl. Akal. Nauk. Rosii, 369(1999),13-15.

25) V. Ryazanov, U. Srebro and E. Yakubov, Plane mappings with dilatations dominated by

functions of bounded mean oscillation, Siberian Adv. Math., 11(2001), 94-130.

26) V. Ryazanov, U Srebro and E. Yakubov, BMO-quasiconformal mappings, J. Anal. Math., 83(2001),1-20.

27) J. Väisälä, Lectures on n-dimensional quasiconformal mappings, Lecture Notes in Math, 229, Springer-Verlag,1971.

28) W.P. Ziemer, Extremal length and p-capacity, Michigan Math. Journal, 16(1969),49-51.

29) V.A. Zoric, On a theorem of M.A. Lavrentiev on quasiconformal mappings in space, Math. Sb., 74(116), (1967), 417-433 (russian).

30) V.A. Zoric, On isolated singularities of mappings with bounded distortion, Math. Sb., 81(123), (1970),634-636 (russian).