FMI, CS, Master I
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Techniques of Combinatorial Optimization
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Seminar 1

(S1.1) A subset $P \subseteq \mathbb{R}^n$ is a polyhedron if and only if $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and some vector $b \in \mathbb{R}^m$.

Proof. Obviously. □

(S1.2) Prove that

(i) Affine sets are polyhedra.

(ii) Singletons are polyhedra of dimension 0.

(iii) Lines are polyhedra of dimension 1.

(iv) The unit cube $C_3 = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1 \text{ for all } i = 1, 2, 3\}$ in $\mathbb{R}^3$ is a full-dimensional polyhedron.

Proof. (i) By Proposition D.0.32.

(ii) Let $b \in \mathbb{R}^n$. Then $\{b\} = b + \{0\}$, hence $\{b\}$ is an affine set of dimension 0.

(iii) Let $x_0, r \in \mathbb{R}^n, r \neq 0$ and $L_{x_0,r}$ be the line through x_0 with direction vector r. Since $L_{x_0,r}$ is affine, it is a polyhedron too. Furthermore, $L_{x_0,r} = x_0 + \text{span}(r)$, hence $L_{x_0,r}$ is an affine set of dimension 1.

(iv) We have that $x \in C_3$ if and only if x is a solution of the system $x \leq \mathbf{1}, -x \leq \mathbf{0}$. Thus, C_3 is a polyhedron. Since $0 \in C_3 \subseteq \text{aff}(C_3)$, it follows that $\text{aff}(C_3)$ is a linear space. Since $e_1, e_2, e_3 \in C_3 \subseteq \text{aff}(C_3)$, we get that $\dim(\text{aff}(C_3)) = 3$.

□

(S1.3) [Farkas lemma - variant] The system $Ax = b$ has a solution $x \geq \mathbf{0}$ if and only if $y^T b \geq 0$ for each $y \in \mathbb{R}^m$ with $y^T A \geq \mathbf{0}^T$.

Proof. Lemma Farkas has the logical form $\neg P \leftrightarrow \exists y(Q(y) \wedge R(y))$, where

$$P \equiv \exists x(Ax = b \wedge x \geq \mathbf{0}), \;\; Q(y) \equiv y^T A \geq \mathbf{0}^T, \;\; R(y) \equiv y^T b < 0.$$

It follows that

$$\begin{aligned} P &\leftrightarrow \neg \exists y(Q(y) \wedge R(y)) \leftrightarrow \forall y \neg(Q(y) \wedge R(y))) \leftrightarrow \forall y(\neg Q(y) \vee \neg R(y)) \\ &\leftrightarrow \forall y(Q(y) \rightarrow \neg R(y)). \end{aligned}$$

□

(S1.4) Let (P) and (D) be the primal and dual LPs.

(i) If both (P) and (D) are feasible, then they are bounded.

(ii) If either (P) or (D) is unfeasible, then the other is either unfeasible or unbounded.

(iii) If either (P) or (D) is unbounded, then the other is unfeasible.

(iv) If either (P) or (D) is bounded, then the other is bounded too.

Proof. (i) By strong duality, Theorem 1.3.4.

(ii) We have two cases:

(a) Assume that (P) is unfeasible. We have to prove that (D) is either unfeasible or unbounded. Suppose that (D) is feasible and let $y \in \mathbb{R}^m$ be a feasible solution, i.e. $y \geq \mathbf{0}$ and $y^T A = c^T$. By Lemma 1.3.2.(i), unfeasibility of (P) implies that there exists $u \in \mathbb{R}^m$ such that $u \geq \mathbf{0}$, $u^T A = \mathbf{0}^T$ and $u^T b < 0$. For any $\lambda \geq 0$, let us denote $y_\lambda = y + \lambda u$. It follows that $y_\lambda \geq \mathbf{0}$ and $y_\lambda^T A = y^T A + \lambda u^T A = c^T$. Thus, y_λ is feasible for (D). Remark that

$$y_\lambda^T b = y^T b + \lambda u^T b \rightarrow -\infty \text{ as } \lambda \rightarrow +\infty, \text{ since } u^T b < 0.$$

It follows that (D) is unbounded.

(b) Assume that (D) is unfeasible. We have to prove that (P) is either unfeasible or unbounded. Suppose that (P) is feasible and let $x \in \mathbb{R}^n$ be a feasible solution, i.e. $Ax \leq b$. By Lemma 1.3.2.(ii), unfeasibility of (D) implies that there exists $u \in \mathbb{R}^m$ such that $Au \geq \mathbf{0}, c^T u < 0$. For any $\lambda \geq 0$, let us denote $x_\lambda = x - \lambda u$. It follows that $Ax_\lambda = Ax - \lambda Au \leq b$, hence x_λ is feasible for (P). Remark that

$$c^T x_\lambda = c^T x - \lambda c^T u \rightarrow +\infty \text{ as } \lambda \rightarrow +\infty, \text{ since } c^T u < 0.$$

It follows that (P) is unbounded.

(iii) By an immediate application of (i).

(iv) By (ii) and (iii).

□