FMI, CS, Master I
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Techniques of Combinatorial
Optimization
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Seminar 2

(S2.1) Let $C_n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1 \text{ for all } i = 1, \ldots, n\}$ be the unit cube in $\mathbb{R}^n$ and F be the intersection of C_n with the hyperplane $\{x \in \mathbb{R}^n \mid x_n = 1\}$. What are the dimensions of C_n and F?

Proof. We apply Corollary 1.4.6. One can easily see that C_n has no implicit equalities. Just take $x = (1/2, 1/2, \ldots, 1/2)^T \in P$. Then $0 \neq x_i \neq 1$ for all $i = 1, \ldots, n$. Thus, C_n is full-dimensional.
Similarly, the only implicit equality of F is $x_n = 1$: just take $x = (1/2, 1/2, \ldots, 1/2, 1)^T \in F$. Then $x_n \neq 0$ and $0 \neq x_i \neq 1$ for all $i = 1, \ldots, n-1$. Thus, $F = \{x \in \mathbb{R}^n \mid x_n = 1, -x_n \leq 0, 0 \leq x_i \leq 1 \text{ for all } i = 1, \ldots, n-1\}$. As a consequence, $A^= = (0, \ldots, 0, 1)$, hence $\dim(F) = n - 1$. □

(S2.2) List all faces of the square $P = \{x \in \mathbb{R}^2 \mid 0 \leq x_i \leq 1 \text{ for } i = 1, 2\}$.

Proof. We have that P is the solution set of a system of 4 inequalities: $-x_1 \leq 0, x_1 \leq 1, -x_2 \leq 0, x_2 \leq 1$, i.e.

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Then $I = \emptyset$ gives $F_I = P$ and

$$\begin{aligned}
I = \{1\} &\Rightarrow F_I = \{x \in P \mid x_1 = 0\} = \{(0, x_2) \mid 0 \le x_2 \le 1\} \\
I = \{2\} &\Rightarrow F_I = \{x \in P \mid x_1 = 1\} = \{(1, x_2) \mid 0 \le x_2 \le 1\} \\
I = \{3\} &\Rightarrow F_I = \{x \in P \mid x_2 = 0\} = \{(x_1, 0) \mid 0 \le x_1 \le 1\} \\
I = \{4\} &\Rightarrow F_I = \{x \in P \mid x_2 = 1\} = \{(x_1, 1) \mid 0 \le x_1 \le 1\} \\
I = \{1,3\} &\Rightarrow F_I = \{x \in P \mid x_1 = 0, x_2 = 0\} = \{(0,0)\} \\
I = \{1,4\} &\Rightarrow F_I = \{x \in P \mid x_1 = 0, x_2 = 1\} = \{(0,1)\} \\
I = \{2,3\} &\Rightarrow F_I = \{x \in P \mid x_1 = 1, x_2 = 0\} = \{(1,0)\} \\
I = \{2,4\} &\Rightarrow F_I = \{x \in P \mid x_1 = 1, x_2 = 1\} = \{(1,1)\}
\end{aligned}$$

For the other I's we get that $F_I = \emptyset$. □

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ be a nonempty polyhedron.

(S2.3) The following are equivalent for every $y \in \mathbb{R}^n, y \neq \mathbf{0}$:

(i) $y \in \text{lin.space}(P)$.

(ii) For every $x \in P$, the line $L_{x,y} \subseteq P$.

(iii) There exists $x \in P$ such that $L_{x,y} \subseteq P$.

Proof. *(i)* $\Rightarrow$ *(ii)* Let $x \in P$. Then $A(x + \lambda y) = Ax + \lambda Ay = Ax \le b$ for every $\lambda \in \mathbb{R}$. Thus, $L_{x,y} \subseteq P$.
(ii) $\Rightarrow$ *(iii)* Obviously.
(iii) $\Rightarrow$ *(i)* Let $x \in P$ be such that $L_{x,y} \subseteq P$. It follows that for all $\lambda \in \mathbb{R}$, $Ax + \lambda Ay \le b$, hence for all $i = 1, \ldots, m$,

$$\mathbf{a}_i x + \lambda \mathbf{a}_i y \le b_i. \tag{1.1}$$

Assume by contradiction that $y \notin \text{lin.space}(P)$. Then there exists $j = 1, \ldots, m$ such that $\mathbf{a}_j y \neq 0$. If $\mathbf{a}_j y > 0$, then $\lim\limits_{\lambda \to \infty} (\mathbf{a}_j x + \lambda \mathbf{a}_j y) = \infty$, contradicting (1.1). If $\mathbf{a}_j y < 0$, then $\lim\limits_{\lambda \to -\infty} (\mathbf{a}_j x + \lambda \mathbf{a}_j y) = \infty$, again contradicting (1.1). □

(S2.4) Let F be a face of a polyhedron P. Then F is again a polyhedron. Furthermore, a subset $F' \subseteq F$ is a face of P if and only if it is face of F.

Proof. We apply Theorem 1.7.5.(iii). Then F is a face of P iff $F = P \cap \{x \in \mathbb{R}^n \mid A_I x = b_I\}$ iff $F = \{x \in \mathbb{R}^n \mid A_I x = b_I, \mathbf{a}_i x \le b_i \text{ for all } i \notin I\}$ iff $F = \{x \in \mathbb{R}^n \mid A_I x \le b_I, -A_I x \le -b_I, \mathbf{a}_i x \le b_i \text{ for all } i \notin I\}$. It follows that F is a polyhedron.

For the second part, let $F' \subseteq F$.
" $\Rightarrow$ " Assume that F' is a face of P, hence $F' = P \cap \{x \in \mathbb{R}^n \mid A_J x = b_J\}$ for some $J \subseteq \{1, \ldots, m\}$. Since $F' \subseteq F \subseteq P$ it is obvious that $F' = F \cap \{x \in \mathbb{R}^n \mid A_J x = b_J\}$, hence F' is a face of F.
" $\Leftarrow$ " Assume that F' is a face of F, so $F' = F \cap \{x \in \mathbb{R}^n \mid A_K x = b_K\}$ for some $K \subseteq \{1, \ldots, m\}$. It follows that $F' = (P \cap \{x \in \mathbb{R}^n \mid A_I x = b_I\}) \cap \{x \in \mathbb{R}^n \mid A_K x = b_K\} = P \cap \{x \in \mathbb{R}^n \mid A_{I \cup K} = b_{I \cup K}\}$. Thus, F' is a face of P. $\square$