FMI, CS, Master I
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Techniques of Combinatorial Optimization
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Seminar 3

(S3.1) Let us consider the cube $C_3 = \{x \in \mathbb{R}^3 \mid 0 \leq x_i \leq 1 \text{ for all } i = 1, 2, 3\}$ in $\mathbb{R}^3$. List the faces, the facets, the minimal faces and the vertices of C_3.

Proof. We have that C_3 is the solution set of a system of 6 inequalities: $-x_1 \leq 0, x_1 \leq 1, -x_2 \leq 0, x_2 \leq 1, -x_3 \leq 0, x_3 \leq 1$, i.e.

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Then $F = \emptyset$ is a face of C_3. The nonempty faces have the form $F_I = \{x \in \mathbb{R}^3 \mid A_I x = b_I\}$ for some $I \subseteq \{1, \ldots, 6\}$. Obviously, $I = \emptyset$ gives $F_I = P$.
The facets of C_3 are:

$$\begin{aligned}
F_{\{1\}} &= \{x \in P \mid x_1 = 0\} = \{(0, x_2, x_3) \mid 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1\} \\
F_{\{2\}} &= \{x \in P \mid x_1 = 1\} = \{(1, x_2, x_3) \mid 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1\} \\
F_{\{3\}} &= \{x \in P \mid x_2 = 0\} = \{(x_1, 0, x_3) \mid 0 \leq x_1 \leq 1, 0 \leq x_3 \leq 1\} \\
F_{\{4\}} &= \{x \in P \mid x_2 = 1\} = \{(x_1, 1, x_3) \mid 0 \leq x_1 \leq 1, 0 \leq x_3 \leq 1\} \\
F_{\{5\}} &= \{x \in P \mid x_3 = 0\} = \{(x_1, x_2, 0) \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\} \\
F_{\{6\}} &= \{x \in P \mid x_3 = 1\} = \{(x_1, x_2, 1) \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}
\end{aligned}$$

The minimal faces of C_3 are:

$$\begin{aligned}
F_{\{1,3,5\}} &= \{x \in P \mid x_1 = 0, x_2 = 0, x_3 = 0\} = \{(0,0,0)\} \\
F_{\{1,3,6\}} &= \{x \in P \mid x_1 = 0, x_2 = 0, x_3 = 1\} = \{(0,0,1)\} \\
F_{\{1,4,5\}} &= \{x \in P \mid x_1 = 0, x_2 = 1, x_3 = 0\} = \{(0,1,0)\} \\
F_{\{1,4,6\}} &= \{x \in P \mid x_1 = 0, x_2 = 1, x_3 = 1\} = \{(0,1,1)\} \\
F_{\{2,3,5\}} &= \{x \in P \mid x_1 = 1, x_2 = 0, x_3 = 0\} = \{(1,0,0)\} \\
F_{\{1,3,6\}} &= \{x \in P \mid x_1 = 1, x_2 = 0, x_3 = 1\} = \{(1,0,1)\} \\
F_{\{1,4,5\}} &= \{x \in P \mid x_1 = 1, x_2 = 1, x_3 = 0\} = \{(1,1,0)\} \\
F_{\{1,4,6\}} &= \{x \in P \mid x_1 = 1, x_2 = 1, x_3 = 1\} = \{(1,1,1)\}
\end{aligned}$$

and the vertices of P are $(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0),$ $(1,0,1), (1,1,0), (1,1,1)$.
The other faces of F are:

$$\begin{aligned}
F_{\{1,3\}} &= \{x \in P \mid x_1 = 0, x_2 = 0\} = \{(0,0,x_3) \mid 0 \leq x_3 \leq 1\} \\
F_{\{1,4\}} &= \{x \in P \mid x_1 = 0, x_2 = 1\} = \{(0,1,x_3) \mid 0 \leq x_3 \leq 1\} \\
F_{\{1,5\}} &= \{x \in P \mid x_1 = 0, x_3 = 0\} = \{(0,x_2,0) \mid 0 \leq x_2 \leq 1\} \\
F_{\{1,6\}} &= \{x \in P \mid x_1 = 0, x_3 = 1\} = \{(0,x_2,1) \mid 0 \leq x_2 \leq 1\} \\
F_{\{2,3\}} &= \{x \in P \mid x_1 = 1, x_2 = 0\} = \{(1,0,x_3) \mid 0 \leq x_3 \leq 1\} \\
F_{\{2,4\}} &= \{x \in P \mid x_1 = 1, x_2 = 1\} = \{(1,1,x_3) \mid 0 \leq x_3 \leq 1\} \\
F_{\{2,5\}} &= \{x \in P \mid x_1 = 1, x_3 = 0\} = \{(1,x_2,0) \mid 0 \leq x_2 \leq 1\} \\
F_{\{2,6\}} &= \{x \in P \mid x_1 = 1, x_3 = 1\} = \{(1,x_2,1) \mid 0 \leq x_2 \leq 1\} \\
F_{\{3,5\}} &= \{x \in P \mid x_2 = 0, x_3 = 0\} = \{(x_1,0,0) \mid 0 \leq x_1 \leq 1\} \\
F_{\{3,6\}} &= \{x \in P \mid x_2 = 0, x_3 = 1\} = \{(x_1,0,1) \mid 0 \leq x_1 \leq 1\} \\
F_{\{4,5\}} &= \{x \in P \mid x_2 = 1, x_3 = 0\} = \{(x_1,1,0) \mid 0 \leq x_1 \leq 1\} \\
F_{\{4,6\}} &= \{x \in P \mid x_2 = 1, x_3 = 1\} = \{(x_1,1,1) \mid 0 \leq x_1 \leq 1\}
\end{aligned}$$

We have that $F_I = \emptyset$ for all other $I \subseteq \{1, \ldots, 6\}$
There are in total 28 faces. □

(S3.2)

(i) If P is an affine set, then its only faces are $\emptyset$ and P.

(ii) If P has proper faces, then $I^+ \neq \emptyset$.

(iii) If F is a proper face of P, then $F = \{x \in P \mid A_I x = b_I\}$ for some $\emptyset \neq I \subseteq I^+$.

Proof. (i) Assume that $P = \{x \in \mathbb{R}^n \mid Ax = b\}$. It follows that

$$P = \{x \in \mathbb{R}^n \mid \mathbf{a}_i x \leq b_i, -\mathbf{a}_i x \leq -b_i \text{ for all } i = 1, \dots, m\} = \{x \in \mathbb{R}^n \mid Cx \leq d\},$$

where $C = \begin{pmatrix} A \\ -A \end{pmatrix}$, $d = \begin{pmatrix} b \\ -b \end{pmatrix}$. Let F be a nonempty face of P. By Theorem 1.7.5.(iii), there exists $I \subseteq \{1, \dots, 2m\}$ such that $F = \{x \in P \mid C_I x = D_I\}$. If $i \in \{1, \dots, m\}$, then $\mathbf{c}_i = \mathbf{a}_i$ and $d_i = b_i$. If $i \in \{m+1, \dots, 2m\}$, then $\mathbf{c}_i = -\mathbf{a}_{i-m}$ and $d_i = -b_{i-m}$. In both cases, $\mathbf{c}_i x = d_i$ for all $x \in P$.

(ii) If $I^+ = \emptyset$, then $P = \{x \in \mathbb{R}^n \mid A^= x = b^=\}$ is affine and has no proper faces, by (i).

(iii) Since $F \neq \emptyset$, by Theorem 1.7.5.(iii), there exists $J \subseteq \{1, \dots, m\}$ such that $F = \{x \in P \mid A_J x = b_J\}$. Take $I := J \cap I^+$. If $I = \emptyset$, then $J \subseteq I^=$, hence $F = P$, that is a contradiction. Thus, we must have $I \neq \emptyset$. Furthermore, for all $x \in P$ we have that $A_I x = b_I$ iff $A_J x = b_J$, since for $i \in J \cap I^=$ one has already that $\mathbf{a}_i x = b_i$ is an implicit equation.

□