FMI, CS, Master I
Winter Term 2014/2015
Techniques of Combinatorial Optimization
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17-18.12.2014

Seminar 4

(S4.1) Let G be the complete graph K_3 on three vertices. Prove that its incidence matrix is not totally unimodular.

Proof. Let $V(K_3) = \{1, 2, 3\}$ and $E(K_3) = \{ij \mid 1 \leq i < j \leq 3\} = \{12, 13, 23\}$. Then its incidence matrix A has 3 rows corresponding to the vertices $1, 2, 3$ and 3 columns, corresponding to the edges $12, 13, 23$. Thus,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We get that $\det(A) = -2$. □

(S4.2) Let $A \in \mathbb{R}^{m\times n}$ be a TU matrix, let $b \in \mathbb{Z}^m$ and let $c \in \mathbb{Z}^n$. Then both optima in the LP duality equation

$$\max\{c^T x \mid x \geq \mathbf{0}, Ax \leq b\} = \min\{y^T b \mid y \geq \mathbf{0}, y^T A \geq c^T\}.$$

have integer optimal solutions (if the optima are finite).

Proof. By Theorem 2.3.3, $P = \{x \mid x \geq \mathbf{0}, Ax \leq b\}$ is integer, so we can apply Proposition 2.2.3.(iv) to conclude that $\max\{c^T x \mid x \geq \mathbf{0}, Ax \leq b\}$ has an integer optimal solution x^*.
Claim: The dual LP is $\min\{y^T b \mid y \geq \mathbf{0}, y^T A \geq c^T\}$.
Proof of Claim: We have that $\max\{c^T x \mid x \geq \mathbf{0}, Ax \leq b\} = \max\{c^T x \mid Cx \leq d\}$, where $C = \begin{pmatrix} -I \\ A \end{pmatrix}$, $d = \begin{pmatrix} \mathbf{0} \\ b \end{pmatrix}$. Thus, its dual is $\min\{w^T d \mid w \geq \mathbf{0}, w^T C = c^T\}$. By letting $w = \begin{pmatrix} u \\ y \end{pmatrix}$, it follows that

$$\begin{aligned} \min\{w^T d \mid w \geq \mathbf{0}, w^T C = c^T\} &= \min\{y^T b \mid u, y \geq \mathbf{0}, -u^T + y^T A = c^T\} \\ &= \min\{y^T b \mid u, y \geq \mathbf{0}, y^T A = c^T + u^T\} \\ &= \min\{y^T b \mid y \geq \mathbf{0}, y^T A \geq c^T\}. \quad \blacksquare \end{aligned}$$

Thus, the dual LP is

$$\min\{y^T b \mid y \geq \mathbf{0}, y^T A \geq c^T\} \;=\; \min\{y^T b \mid y \geq \mathbf{0}, A^T y \geq c\} = \min\{b^T y \mid Dy \leq a\},$$

where $D = \begin{pmatrix} -I \\ -A^T \end{pmatrix}$ and $a = \begin{pmatrix} \mathbf{0} \\ -c \end{pmatrix}$. Since D is obtained from A by using operations that preserve the TU property, C is also a TU matrix. As a is an integer vector, the dual polyhedron is integer and the dual LP has an integral optimal solution y^*. □

(S4.3) Let $G = (V, E)$ be a bipartite graph and $w : E \to \mathbb{N}$ be a weight function. The maximum weight of a matching in G is equal to the minimum value of $\sum_{v \in V} y_v$, where y ranges over all functions $y : V \to \mathbb{N}$ such that $y_u + y_v \geq w(e)$ for each edge $e = uv$ of G.

Proof. We have that

$$\begin{aligned} \max\{w(M) \mid M \text{ matching in } G\} &= \max\{w^T x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} \\ &= \min\{y^T \mathbf{1} \mid y \geq \mathbf{0}, y^T A \geq w^T\} \\ &= \min\{y^T \mathbf{1} \mid y \geq \mathbf{0}, y^T A \geq w^T, y \in \mathbb{Z}^V\}, \end{aligned}$$

as a consequence of Proposition 2.3.4. Since $y \in \mathbb{Z}^V$ and $y \geq \mathbf{0}$, we get that $y \in \mathbb{N}^V$, hence $y : V \to \mathbb{N}$. Remark that $y^T \mathbf{1} = \sum_{v \in V} y_v$ and that for every edge $e = uv$ of G, $(y^T A)_e = y_u + y_v$. Thus, $y^T A \geq w^T$ if and only if $y_u + y_v \geq w(e)$ for each edge $e = uv$ of G. □

(S4.4) Let A be the incidence matrix of a cycle of length 5. Prove that A is a square matrix and compute its determinant.

Proof. Let $C_5 = v_0 v_1 v_2 v_3 v_4$ be the cycle of length 5, with vertices v_0, v_1, v_2, v_3, v_4 and edges $v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_0$. Then its incidence matrix is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

One can easily see that $det(A) = 2$. □

(S4.5) Give examples of nonempty polyhedra $P \subseteq \mathbb{R}^n$ such that

(i) $P \cap \mathbb{Z}^n = \emptyset$, hence $P_I = \emptyset$.

(ii) $P \cap \mathbb{Z}^n \neq \emptyset$ and for every $c \in \mathbb{R}^n \setminus \{0\}$, $\max\{c^T x \mid x \in P \cap \mathbb{Z}^n\} < \max\{c^T x \mid x \in P\}$.

Proof. (i) Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and define $P := \{(\alpha, \ldots, \alpha)^T\}$. Then $P \cap \mathbb{Z}^n = \emptyset$. Or, take $P := [1/3, 1/2]$.

(ii) For example, take $P := [5/2, 7/2]$. Then $P \cap \mathbb{Z} = \{3\}$, hence $\max\{c^T x \mid x \in P \cap \mathbb{Z}^n\} = max\{3c\} = 3c$. On the other hand, for every $c \in \mathbb{R} \setminus \{0\}$, we have that

(a) if $c > 0$, then $\max\{c^T x \mid x \in P\} = 7c/2 \neq 3c$,

(b) if $c < 0$, then $\max\{c^T x \mid x \in P\} = 5c/2 \neq 3c$.

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