FMI, CS, Master I
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Techniques of Combinatorial Optimization
Laurenţiu Leuştean

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Seminar 6

In the sequel, $D = (V, A)$ is a digraph, $s, t \in V$.

(S6.1) Assume that $f_1, \ldots, f_n : A \to \mathbb{R}$ are mappings satisfying the flow conservation law at $v \in V$. Then any linear combination of $f_1, \ldots, f_n$ satisfies the flow conservation law at v.

Proof. Let $\lambda_i \in \mathbb{R}, i = 1, \ldots, n$ and $f := \sum_{i=1}^{n} \lambda_i f_i$. Then for all $v \in V$,

$$in_f(v) = \sum_{a \in \delta^{in}(v)} f(a) = \sum_{a \in \delta^{in}(v)} \sum_{i=1}^{n} \lambda_i f_i(a) = \sum_{i=1}^{n} \lambda_i \left(\sum_{a \in \delta^{in}(v)} f_i(a) \right) = \sum_{i=1}^{n} \lambda_i in_{f_i}(v)$$

and similarly

$$out_f(v) = \sum_{i=1}^{n} \lambda_i out_{f_i}(v).$$

Apply now the fact that $in_{f_i}(v) = out_{f_i}(v)$ for all $i = 1, \ldots, n$ to conclude that $in_f(v) = out_f(v)$. □

(S6.2) Prove Proposition 3.4.2.

Proof. Apply the Flow Decomposition Theorem 3.4.1. Then there exist $K, L \in \mathbb{Z}_+$, positive numbers $w_1, \ldots, w_K, \mu_1, \ldots, \mu_L$, s-t paths $P_1, \ldots, P_K$ and circuits $C_1, \ldots, C_L$ such that

$$f = \sum_{i=1}^{K} w_i \chi^{P_i} + \sum_{j=1}^{L} \mu_j \chi^{C_j} \quad \text{and} \quad \text{value}(f) = \sum_{i=1}^{K} w_i.$$

Furthermore, the w_i's, μ_j's are positive integers. Since f is a $\{0,1\}$-flow, we must have $w_i = \mu_j = 1$ for all i, j. Thus,

$$f = \sum_{i=1}^{K} \chi^{P_i} + \sum_{j=1}^{L} \chi^{C_j} \quad \text{and} \quad \text{value}(f) = K.$$

It remains to show that the family $\mathcal{F} = \{P_1, \ldots, P_K, C_1, \ldots, C_L\}$ is arc-disjoint. If $Q_1, Q_2 \in \mathcal{F}$ have an arc a in common, then $f(a) \geq \chi^{Q_1}(a) + \chi^{Q_2}(a) = 2$, which contradicts the fact that f is a $\{0,1\}$-flow. □

(S6.3) For any s-t path P in D, prove that χ^P satisfies the flow conservation law at every $v \neq s, t$ and that value$(\chi^P) = 1$.

Proof. If $P = st$, then $\chi^P(a) = 0$ for all $a \neq (s,t)$, hence $in_{\chi^P}(v) = out_{\chi^P}(v) = 0$ for all $v \neq s, t$. Assume that $P = sv_1 \ldots v_k t$ with $k \geq 1$. Let us denote $v_0 := s, v_{k+1} := t$. Then $\chi^P((s,v_1)) = \chi^P((v_1,v_2)) = \ldots = \chi^P((v_{k-1},v_k)) = \chi^P((v_k,t)) = 1$ and $\chi^P(a) = 0$ for all the other arcs a. For an arbitrary $v \neq s, t$ we have two cases:

(i) $v \notin P$. Then $in_{\chi^P}(v) = out_{\chi^P}(v) = 0$.

(ii) $v = v_i, i = 1, \ldots, k$. Then

$$out_{\chi^P}(v_i) = \sum_{a \in \delta^{in}(v_i)} \chi^P(a) = \chi^P((v_{i-1}, v_i)) + 0 = 1,$$
$$out_{\chi^P}(v_i) = \sum_{a \in \delta^{out}(v_i)} \chi^P(a) = \chi^P((v_i, v_{i+1})) + 0 = 1.$$

Finally,

$$\text{value}(\chi^P) = out_{\chi^P}(s) - in_{\chi^P}(s) = \chi^P((s,v_1)) - 0 = 1.$$

□

(S6.4) Let $N = (D, s, t)$ be a unit capacity network, $k \geq 1$ and $P_1, \ldots, P_k$ be k arc-disjoint s-t paths in D. Then for all $k \geq 1$,

$$f := \chi^{P_1} + \ldots + \chi^{P_k}$$

is an s-t $\{0,1\}$-flow f with value$(f) = k$.

Proof. For $k = 1$, we have that $f := \chi^{P_1}$ is an s-t flow of value 1, by (S6.3) and the fact that $0 \leq \chi^{P_1} \leq 1$.
Let $k \geq 2$. Then f satisfies the flow conservation law at every $v \neq s, t$, by (S6.3) and (S6.1), and value$(f) = k$. It remains to prove that f takes values in $\{0, 1\}$. For any $a \in A$, we have one of the two cases:

(i) $a \notin P_1 \cup \ldots \cup P_k$, so $f(a) = 0$.

(ii) there exists a unique $i = 1, \ldots, k$ such that $a \in P_i$, so $f(a) = 1$.

Thus, $f : A \to \{0, 1\}$ is a flow. □

(S6.5) Let us recall that a subset $B \subseteq A$ is said to be an s-t *disconnecting arc set* if B intersects each s-t path. Prove that each s-t cut is an s-t disconnecting arc set.

Proof. Let $\delta^{out}(U)$ be an s-t cut, where $s \in U$ and $t \notin U$. Let $P = sv_1 \ldots v_k t$ be an s-t path and denote $v_0 := s$ and $v_{k+1} := t$. We have two cases:

(i) there exists $i = 1, \ldots, k$ such that $v_i \notin U$ and $v_{i-1} \in U$. Then $(v_{i-1}, v_i) \in \delta^{out}(U)$.

(ii) $v_i \in U$ for all $i = 1, \ldots, k$. Then $(v_k, t) \in \delta^{out}(U)$.

□