FMI, CS, Master I
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Techniques of Combinatorial Optimization
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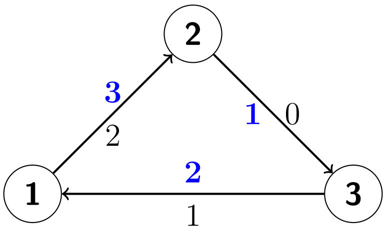
Seminar 7

(S7.1) Give an example where feasible circulations do not exist.

Proof. We consider the following example



Thus,

(i) $D = (V, A)$, where $V = \{1, 2, 3\}, A = \{(1, 2), (2, 3), (3, 1)\}$,

(ii) $c((1, 2)) = 3, c((2, 3)) = 1, c((3, 1)) = 2$ and

(iii) $d((1, 2)) = 2, d((2, 3)) = 0, d((3, 1)) = 1$.

Let $f : A \to \mathbb{R}$ be feasible w.r.t. d, c. Then $in_f(2) = f((1, 2)) \geq d((1, 2)) = 2$, while $out_f(2) = f((2, 3)) \leq c((2, 3)) = 1$. Thus, $in_f(2) \neq out_f(2)$, so f cannot be a circulation. $\square$

(S7.2) Prove Menger's theorem (directed vertex-disjoint version): If $D = (V, A)$ is a digraph and $S, T \subseteq V$, then the maximum number of vertex-disjoint S-T-paths is equal to the minimum size of an S-T disconnecting vertex set.
(This is Theorem 4.1.10 in the lecture notes).

Proof. Make a digraph D' as follows from D: add two new vertices s and t and arcs (s,v) for all $v \in S$ and (v,t) for all $v \in T$. Thus,

$$V(D') = V \cup \{s,t\} \text{ and } A(D') = A \cup \{(s,v) \mid v \in S\} \cup \{(v,t) \mid v \in T\}.$$

Then s and t are nonadjacent in D'. If $P = v_1 \dots v_k$ is an S-T path in D, then obviously $P' := sv_1 \dots v_k t$ is an s-t-path in D'. Furthermore, any s-t-path in D' is of the form P' for some s-t path P in D. It follows that two S-T-paths P, Q in D are vertex-disjoint in D if and only if the s-t-paths P', Q' are internally vertex-disjoint in D'.
Claim: s-t vertex-cuts in D' coincide with S-T disconnecting vertex sets in D.
Proof of Claim: Assume that U is an s-t vertex-cut in D'. Then $s, t \notin U$, so $U \subseteq V$. If $v_1 \dots v_k$ is an S-T-path in D, then $sv_1 \dots v_k t$ is an s-t-path in D', so it intersects U. Thus, $v_i \in U$ for some $i = 1, \dots, k$.
Conversely, assume that U is an S-T disconnecting vertex set in D and let $sv_1 \dots v_k t$ be an s-t-path in D'. Then $v_1 \dots v_k$ is an S-T-path in D, hence there exists $i = 1, \dots, k$ such that $v_i \in U$. ■
Then Theorem 4.1.9 applied to D', s, t concludes the proof. □

(S7.3) Let $D = (V, A)$ and f', f be feasible circulations in D. Define $g : A \cup A^{-1} \to \mathbb{R}$ as follows: for all $a \in A$,

$$g(a) = \max\{0, f'(a) - f(a)\}, \quad g(a^{-1}) = \max\{0, f(a) - f'(a)\}.$$

Prove that

(i) g is a circulation in $\overline{D}$;

(ii) $\mathrm{cost}(g) = \mathrm{cost}(f') - \mathrm{cost}(f)$;

(iii) $g(e) = 0$ for all $e \notin A(D_f)$.

(This is Lemma 3.6.3 in the lecture notes).

Proof. **Claim:** $g(a) - g(a^{-1}) = f'(a) - f(a)$ for all $a \in A$.
Proof of Claim: Let $a \in A$. We have two cases:

(i) $f'(a) \geq f(a)$. Then $g(a) = f'(a) - f(a)$ and $g(a^{-1}) = 0$.

(ii) $f'(a) < f(a)$. Then $g(a) = 0$ and $g(a^{-1}) = f(a) - f'(a)$.

■

(i) We get that

$$\begin{aligned} in_g(v) &= \sum_{e\in\delta^{in}_{\overline{D}}(v)} g(e) = \sum_{a\in\delta^{in}_A(v)} g(a) + \sum_{a\in\delta^{out}_A(v)} g(a^{-1}) \\ in_g(v) &= \sum_{e\in\delta^{out}_{\overline{D}}(v)} g(e) = \sum_{a\in\delta^{out}_A(v)} g(a) + \sum_{a\in\delta^{in}_A(v)} g(a^{-1}). \end{aligned}$$

Thus,

$$\begin{aligned} out_g(v) - in_g(v) &= \sum_{a\in\delta^{out}_A(v)} (g(a) - g(a^{-1})) - \sum_{a\in\delta^{in}_A(v)} (g(a) - g(a^{-1})) \\ &= \sum_{a\in\delta^{out}_A(v)} (f'(a) - f(a)) - \sum_{a\in\delta^{in}_A(v)} (f'(a) - f(a)) \\ &= out_{f'}(v) - out_f(v) - in_{f'}(v) + in_f(v) = 0. \end{aligned}$$

since f and f' are circulations.

(ii) We have that

$$\begin{aligned} \mathrm{cost}(g) &= \sum_{e\in A\cup A^{-1}} k(a)g(a) = \sum_{a\in A} k(a)g(a) + \sum_{a\in A} k(a^{-1})g(a^{-1}) \\ &= \sum_{a\in A} k(a)(g(a) - g(a^{-1})) = \sum_{a\in A} k(a)(f'(a) - f(a)) = \mathrm{cost}(f') - \mathrm{cost}(f). \end{aligned}$$

(iii) Let $e \notin A(D_f)$. We have two cases:

(a) $e = a \in A$. Then $c(a) = f(a)$, so $f'(a) \leq f(a)$. It follows that $g(e) = g(a) = 0$.

(b) $e = a^{-1}, a \in A$. Then $d(a) = f(a)$, so $f'(a) \geq f(a)$. It follows that $g(e) = g(a^{-1}) = 0$.

□