

# Rewrite dilation structures: a theory-metattheory correspondence

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Continues after [2] [Why approximate sum is composition](#) and after the post [3] [Baby steps, important ideas: rewrite dilation structures](#) where I tried to teach Qwen to obtain proofs in dilation structures from chains of beta reductions in untyped lambda beta calculus.

**A theory-metattheory correspondence.** The point is that there is a correspondence between lambda calculus terms and dilation constructs, like for example that the composition combinator corresponds to the approximate sum or [4] [Why the S combinator is curvature](#). This correspondence is even deeper. There is a way to transform chains of beta reductions of lambda terms into proofs in dilation structures.

This is unlike a Curry-Howard correspondence. Recall that under the Curry-Howard isomorphism, terms in lambda calculus correspond to proofs, types correspond to propositions, and beta-equivalence corresponds to proof simplification (cut elimination).

Therefore in a CH correspondence each lambda term corresponds to a proof and a chain of beta reductions transform this proof into another proof. In the language I use since the introduction of the distinction NTC vs TC, see the tag [5] [NTC vs TC](#), if we assimilate proofs with computations then it appears that the beta reduction does not compute. This is extremely weird in my opinion and it is a legacy of the way logicians approached computation before computers. Last time discussed in [6] [The theory-metattheory trick](#).

This correspondence between chains of beta reductions and proofs in dilation structures is therefore different and computation friendly. This time a construct in dilation structures (the analogue of a lambda term) is not a proof, it is state. The proof in dilation structures can be deduced from a chain of beta reductions, therefore beta reductions compute. This justifies the name "theory-metattheory correspondence".

Here is how we can do this.

Recall that we connect lambda calculus with dilation structures by the following notation, imported from [1] [Pure See](#): we use a formal parameter  $z$  (which will generate the group of integer powers of  $z$ ) and we define the following dilation with coefficients related to  $z$ :

$$\delta_{1/z}^b a = ab$$

$$\delta_z^x a = \lambda x.a$$

**Standard beta rule:**  $(\lambda x.a) b \longrightarrow a[x := b]$

The beta rule in the dilation-structure-friendly notation reads:  $\delta_{1/z}^b (\delta_z^x a) \longrightarrow a[x := b]$ . With  $\varepsilon = 1/z$  we discover a beta like rewrite which transforms a dilation structure into a rewrite dilation structure.

**Dilations beta rule:**  $\delta_\varepsilon^b (\delta_{1/\varepsilon}^x a) \longrightarrow a[x := b]$ . Moreover, we use this rule for writing dilation structures proofs, in the following form: let  $x = b$  in  $c = \delta_\varepsilon^b \delta_{1/\varepsilon}^x a$ . Then  $c = a$ .

You can see in [3] how Qwen succeeds into proving the approximate associativity of approximate sums from the associativity of function composition (expressed with with the B combinator). Here we see another proof, which also solves a problem encountered before.

Indeed, we saw that the approximate sum

$$\Sigma_\varepsilon^e(a, b) = \delta_{1/\varepsilon}^e \delta_\varepsilon^e a b \quad (1)$$

corresponds to the composition combinator  $B = \lambda x. \lambda y. \lambda e. (x (y e))$ , in the sense that

$$B b a = \lambda e. (b (a e))$$

is the lambda term which translates into the approximate sum (1).

The approximate difference does not correspond to a lambda term directly. Recall that the approximate difference is defined by:

$$\Delta_\varepsilon^e(a, b) = \delta_{1/\varepsilon}^{\delta_\varepsilon^e a} \delta_\varepsilon^e b \quad (2)$$

Under  $z = 1/\varepsilon$  the approximate difference becomes the following expression, which is not a lambda term:

$$\delta_z^{\delta_{1/z}^e a} \delta_{1/z}^e b = \lambda(a e). (b e)$$

The same phenomenon happens with the approximate inverse:

$$\text{inv}_\varepsilon^e a = \delta_{1/\varepsilon}^{\delta_\varepsilon^e a} e \quad (3)$$

Under  $z = 1/\varepsilon$  this dilations construct would become the expression

$$\delta_z^{\delta_{1/z}^e a} e = \lambda(a e). e$$

which is not a lambda term. What can we do?

In terms of dilation structures we can reformulate the approximate difference as a composition between approximate sum and approximate inverse

$$\Delta_\varepsilon^e(a, b) = \Sigma_\varepsilon^{\delta_\varepsilon^e a} (\text{inv}_\varepsilon^e a, b)$$

(which we shall not justify here, for brevity) therefore we can concentrate our attention to solve the correspondence problem for the approximate inverse only. Instead of proving that

$$\Sigma_\varepsilon^e(a, \Delta_\varepsilon^e(a, b)) = b$$

we use the previous formula and we try to prove that

$$\Sigma_\varepsilon^e(a, \Sigma_\varepsilon^{\delta_\varepsilon^e a} (\text{inv}_\varepsilon^e a, b)) = \delta_{1/\varepsilon}^e \delta_\varepsilon^e b \quad (4)$$

by using rewrite dilation structures.

For this we start from a different dilation construct, namely

$$\Sigma_\varepsilon^e(a, \Sigma_\varepsilon^u(\delta_{1/\varepsilon}^v e, b)) \quad (5)$$

which translates perfectly into the lambda term:

$$T = \lambda e.((\lambda u.(b((\lambda v.e) u))) (a e)) \quad (6)$$

**Full combinator conversion.** Remark that by using the eta optimized [7] [Kiselyov combinator translation](#) it turns out that

$$T = S (B (B b) K) a$$

and the combinator

$$\lambda b. \lambda a. T = B S (R K (B B B))$$

where  $R a b c = b c a$ . See the related [8] [Abstraction elimination, knots and chora](#).

By two beta reductions the term  $T$  from (6) transforms into  $\lambda e.(b e)$ , which corresponds to the right hand side of the dilation structure equality (4). Moreover the two beta reductions can be translated into dilation beta reductions and they produce the need to identify:  $v = u$  and  $u = \delta_\varepsilon^e a$ . When we plug these identifications into the rewrite dilation structure construct (5) we recover the left hand side of the equality we want to prove.

Let us see this in detail.

**Step 1.** In the term  $T$  from (6) we beta reduce the redex  $((\lambda v.e) u)$  and we obtain the identification  $v = u$  and the new term

$$T = \lambda e.((\lambda u.(b e)) (a e)) \quad (7)$$

At the level of dilation structures this translates as: Let  $v = u$  in the dilations construct (5), where we use the definition of approximate sum (1) to express it as a dilations expression, ie:

$$T = \delta_{1/\varepsilon}^e \delta_\varepsilon^e a \delta_{1/\varepsilon}^u \delta_\varepsilon^u \delta_{1/\varepsilon}^e b$$

Then we use  $\delta_\varepsilon^u \delta_{1/\varepsilon}^u e = e$  to rewrite:

$$T = \delta_{1/\varepsilon}^e \delta_\varepsilon^e a \delta_{1/\varepsilon}^u \delta_\varepsilon^e b \quad (8)$$

**Step 2.** In the term (7) we beta reduce the redex  $((\lambda u.(b e)) (a e))$  and we obtain the identification  $u = a e$  and the new term

$$T = \lambda e.(b e) \quad (9)$$

At the level of dilation structures this translates as: Let  $u = \delta_\varepsilon^e a$  in the dilations construct (8), ie:

$$T = \delta_{1/\varepsilon}^e \delta_\varepsilon^e a \delta_{1/\varepsilon}^e a \delta_\varepsilon^e b$$

Then we use

$$\delta_\varepsilon^e a \delta_{1/\varepsilon}^e a \delta_\varepsilon^e b = \delta_\varepsilon^e b$$

to rewrite:

$$T = \delta_{1/\varepsilon}^e \delta_\varepsilon^e b \quad (10)$$

**Step 3.** We thus proved that

$$\Sigma_\varepsilon^e(a, \Sigma_\varepsilon^{\delta_\varepsilon^e a}(\delta_{1/\varepsilon}^{\delta_\varepsilon^e a} e, b)) = \delta_{1/\varepsilon}^e \delta_\varepsilon^e b$$

This is exactly our goal (4), after we use the definition of approximate inverse (3).

We could also try another approach, by using the approximate associativity we already have, namely first use

$$\Sigma_\varepsilon^e(a, \Sigma_\varepsilon^{\delta_\varepsilon^e a}(\text{inv}_\varepsilon^e a, b)) = \Sigma_\varepsilon^e(\Sigma_\varepsilon^e(a, \text{inv}_\varepsilon^e a), b)$$

and then simply try to prove that

$$\Sigma_\varepsilon^e(a, \text{inv}_\varepsilon^e a) = \delta_{1/\varepsilon}^e e$$

This last equality can be obtained from the rewrite dilation structure construct

$$\Sigma_\varepsilon^e(a, \delta_{1/\varepsilon}^v e)$$

This corresponds to the term

$$T' = \lambda e.((\lambda v.e)(ae))$$

which beta reduces to  $\lambda e.e$  (ie the right hand side in the correspondence) and suggests the identification  $v = ae$ . This identification, translated back into dilation structures notation, allows to transform the rewrite dilation structure construct into the left hand side dilation structure expression.

The last identity from [9] [Emergent algebras tetrad](#) can be proved that it comes from the correspondence with the term

$$\lambda e.((\lambda v.(ae))((\lambda u.(b(au)))e))$$

which beta reduces to  $\lambda e.(ae)$ . By using the same eta optimized Kiselyov combinator translation [7], it turns out that

$$\lambda b.\lambda a.\lambda e.((\lambda v.(ae))((\lambda u.(b(au)))e)) = B(S(BS(BK)))B$$

## References

- [1] M. Buliga. [Introduction to Pure See](#)
- [2] M. Buliga (chorasimilarity). [Why approximate sum is composition](#)
- [3] M. Buliga (chorasimilarity). [Baby steps, important ideas: rewrite dilation structures](#)
- [4] M. Buliga (chorasimilarity). [Why the S combinator is curvature](#)
- [5] M. Buliga (chorasimilarity). [NTC vs TC](#)
- [6] M. Buliga (chorasimilarity). [The theory-metattheory trick](#)
- [7] Ben Lynn. [Kiselyov combinator translation](#)
- [8] M. Buliga (chorasimilarity). [Abstraction elimination, knots and chora](#)
- [9] M. Buliga (chorasimilarity). [Emergent algebras tetrad](#)