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A symplectic Brezis–Ekeland–Nayroles principle

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Abstract
We propose a modification of the Hamiltonian formalism which can be used for dissipative systems. This work continues a previous work on Hamiltonian inclusions and advances by the introduction of a symplectic version of the Brezis–Ekeland–Nayroles principle. As an application we show how standard plasticity can be treated in our formalism.

Keywords
Hamiltonian methods, Brezis–Ekeland–Nayroles principle, convex dissipation, standard plasticity.

1. Introduction

Realistic dynamical systems considered by engineers and physicists are subjected to energy loss. It may ensue from external actions, in which case we call them 'non-conservative'. On the other hand, if the cause is internal, resulting from a broad spectrum of phenomena such as collisions, surface friction, viscosity, plasticity, fracture, damage and so on, we name them 'dissipative'.

However, at least in the first approximation, dissipation may often be neglected and the system can be considered as idealized, which allows applying the methods of classical dynamics. The positions of the bodies are described by degrees of freedom x^i which can be viewed as coordinates of a point x in the system's configuration space X, and the motion is governed by a second-order system of ordinary differential equations (ODEs). A fruitful idea is to break the equations by considering the corresponding momenta y_i as coordinates of a point y living in the space Y, which leads to work in the phase space $X \times Y$ of points $z = (x, y)$. Hence the motion is governed by the first-order system of canonical equations generated by the Hamiltonian function H, which allows applying the very wide range of mathematical methods devoted to smooth functions to solve such problems.

In modern presentation of the topic, the differential geometry offers a powerful framework to tackle the problems from a global viewpoint. For instance, using suitable canonical transformations (i.e. coordinate changes in the phase space preserving the structure of canonical equations) that force the Hamiltonian to be zero leads to trivial solutions in the new coordinates which solve the problem when expressed in the initial ones. Other tools are the symplectic form ω (an antisymmetric tensor of rank two, or 2-form) and the Hamiltonian vector field XH (the symplectic gradient of the Hamiltonian) allowing us to recast the canonical equations in a more compact way. Many such techniques of classical dynamics are prototypes for infinite-dimensional systems, for example

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ideal fluids of which the configuration space is one of the volume-preserving diffeomorphisms. Among the other tools to solve dynamics problems, it is worth quoting the integral of motions and, in the modern language of Lie group theory, the momentum map linked to Noether's theorem [1].

On the other hand, deformations of solids and motions of fluids are modelled through constitutive laws. Due to collisions, brittle fracture and threshold effects, most dissipative laws are non-smooth and multivalued. Moreover, experimental testing suggests that convexity is a keystone property of these phenomenological laws. In the framework of convex analysis, the usual gradient can be generalized thanks to the subdifferential $\partial\phi(z)$ (a set of subgradients of a convex and lower semicontinuous function ϕ at a given point z). Generalized standard materials [2] can be represented by a potential function ϕ, convex but not differentiable everywhere, naturally leading to variational methods and unconstrained or constrained optimization problems with the numerical simulations in the background. Among them, the Brezis–Ekeland–Nayroles (BEN) principle [3–5] is based on the time integration of the sum of dissipation potential ϕ and its Fenchel polar (analogous to the Legendre polar for convex functions). Although little used in the literature, this principle is noteworthy in the sense that it allows covering the whole evolution of the dissipative system.

Classical dynamics is generally addressed through the world of smooth functions while the mechanics of dissipative systems deals with the one of non-smooth functions. Unfortunately, both worlds widely ignore each other. The aim of this work is to lay strong foundations for bridging both worlds and their corresponding topics. In a previous paper [6], Buliga proposed the formalism of Hamiltonian inclusions, able to model dynamical systems with 1-homogeneous dissipation potential (for laws such as brittle damage using the Ambrosio–Tortorelli functional [7]). This formalism is a dynamical version of the quasistatic theory of rate-independent systems of Mielke [8–10].

The key idea is to additively decompose the time rate $\dot{z}$ into reversible part $\dot{z}_R$ (the symplectic gradient) and dissipative or irreversible one $\dot{z}_I$, and next to define the symplectic subdifferential $\partial^{\omega}\phi(z)$ of the dissipation potential. To release the restrictive hypothesis of 1-homogeneity (in particular to address viscoplasticity), we introduce in this work the symplectic Fenchel polar $\phi^{*\omega}$, which allows us to generalize the Hamiltonian inclusion formalism by combining it with the BEN principle.

Our aim is to build theoretical methods to model and analyse dynamical dissipative systems in a consistent geometrical framework with the numerical approaches not very far in the background. The objective is threefold.

- *Extending to the dissipative systems the geometrical methods of classical dynamics by a suitable change of coordinates*. The reference reversible system governed by the Hamiltonian can be described by reversible coordinates. The true dissipative system whose behaviour is more complex than that of the reference system could be modelled with extra dissipative coordinates. Action-angle coordinates are a classical example useful in the case of autonomous systems, for instance to find oscillation frequencies without solving the equation of motion. For non-autonomous systems, Zhuravlev's invariant normalization [11] allows us to reduce the Hamiltonian to normal Birkhoff form. In this respect, the use of Lie group theory and the momentum map is valuable for analysing the symmetries of the problems and guessing appropriate coordinates in which the problem is simpler [1].
- *Exploring the dissipative rheological models in dynamical situations*. Such models are widely developed in statics but their identification in dynamics is making quick progress thanks to improvements in the experimental testing. Nevertheless, we believe that the help of a consistent theoretical framework may be welcome for this task.
- *Using the dynamical BEN principle to solve evolution problems*. Indeed, using the step-by-step numerical methods widely prevalent today, the errors grow with the step number and the integration fails if it does not converge at a given step (difficulty to restart). In contrast, the BEN principle allows us to have a consistent view of the whole evolution by determining all the steps simultaneously. On the other hand, solving space-time principles is more time-consuming but could be improved using model reduction methods such as proper generalized decomposition (PGD) [12–15].

Closer to the subject of this article, we cite the contributions of Aubin [16], Aubin et al. [17], and Rockafellar [18], which considered various extensions of Hamiltonian and Lagrangian mechanics. In the article by Bloch et al. [19], Hamiltonian systems are explored with an added Rayleigh dissipation. A theory of quasistatic rate-independent systems is proposed by Mielke and Theil [10], and Mielke [8], and developed for applications in many papers, among them Mielke and Roubíček [9]; see also the very recent paper by Visintin [20]. In [21, 22], Grmela and Öttinger proposed the framework GENERIC (General Equation for Non-Equilibrium Reversible-Irreversible Coupling), a systematic method to derive thermodynamically consistent evolution equations. A variational formulation of GENERIC is proposed in [23]. In [24], Mielke proposed a GENERIC formulation for generalized standard materials quite similar to the one in the present paper. It would be worth making the present formalism thermodynamic. In [25], Stefanelli used the BEN variational principle to represent the quasistatic evolution of an elastoplastic material with hardening in order to prove the convergence of time and space-time discretizations as well as to provide some possible a posteriori error control. Finally, Ghoussoub and MacCann characterized the path of steepest descent of a non-convex potential as the global minimum of the BEN functional [26].

Moreover, another advantage of the BEN principle is ease of generalization. Indeed, it is worth knowing that many realistic dissipative laws, called 'non-associated', cannot be cast in the mould of the standard ones deriving from a dissipation potential. To skirt this pitfall, in [27] De Saxcé and Feng proposed a new theory based on a function called 'bipotential'. It physically represents the dissipation and generalizes the sum of the dissipation potential and its Fenchel polar, a reason why extension of the BEN principle is natural.

The applications to solid mechanics are varied: Coulomb's friction law [28], the non-associated Drücker–Prager [29] and Cam-Clay models [30] in soil mechanics, cyclic plasticity [28, 31] and viscoplasticity [32] of metals with the non-linear kinematical hardening rule, Lemaitre's damage law [33], and the coaxial laws [34, 35]. Such kinds of materials are called implicit standard materials and define a very broad extension of the generalized standard materials. A synthetic review of these laws can be found in [34, 35]. It is also worth noting that monotone laws which do not admit a convex potential can nevertheless be represented by Fitzpatrick's function [36]. In [37], we showed that bipotentials are related to Fitzpatrick functions associated to a maximally monotone operator. Recently, Visintin proposed the extension of the Brezis–Ekeland principle to monotone operators [38]. Mielke et al. have developed an extended theory concerning various nonlinear dissipative problems also based on bipotentials [39]. The self-dual Lagrangians introduced and studied by Ghoussoub [40, 41] and Ghoussoub and Moameni [42] can also be seen as bipotentials representing maximal monotone operators.

The paper is organized as follows.

- In Section 2, we redefine the familiar notions of subdifferential, gradient, and Fenchel transform, by using a symplectic form instead of the usual duality.
- Section 3 is essentially a reminder of the Hamiltonian systems in order to introduce dissipative ones.
- Section 4 is the heart of the paper. The cornerstone idea is to decompose the velocity in the phase space into reversible and irreversible parts. The evolution of the dissipative system is governed by two functions, the Hamiltonian and a dissipation convex potential. The second idea is to propose a variational version extending the BEN principle to dynamics on the basis of the notions introduced in Section 2. In a geometrical spirit, we discussed its invariance. Variational inequations are deduced from this principle, in particular by associating to an arbitrary smooth function a variation through its symplectic gradient.
- Section 5 is devoted to the application to standard plasticity in order to illustrate the proposed formulation.
- In Section 6, some perspectives are discussed concerning new numerical algorithms based on the present variational formulation and the extension of geometrical methods of Hamiltonian systems to dissipative ones.

2. Preliminaries and notations

In this section we redefine the familiar notions of subdifferential, gradient, and Fenchel transform, by using a symplectic form instead of the usual duality.

We start with the following setting (but see Remark 2.5 for possible generalizations). X and Y are topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$. There is a duality product

$$\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$$

such that any continuous linear functional on X (resp. on Y) has the form $x \mapsto \langle x, y \rangle$, for some $y \in Y$ (resp. $y \mapsto \langle x, y \rangle$, for some $x \in X$).

For a general element of $X \times Y$ we shall use the notation $z = (x, y)$, or similar.

The space $X \times Y$ has a natural symplectic (i.e. bilinear and antisymmetric) form $\omega : (X \times Y)^2 \to \mathbb{R}$ which is defined via the duality product by the following formula: for any $z = (x, y)$ and $z' = (x', y')$

$$\omega(z, z') = \langle x, y' \rangle - \langle x', y \rangle.$$

Definition 2.1 *The symplectic subdifferential of $F : X \times Y \to \mathbb{R} \cup \{+\infty\}$, a convex lower semicontinuous (lsc) function, is the function which associates to any $z = (x, y) \in X \times Y$ such that for $F(z) < +\infty$ the set*

$$\partial^\omega F(z) = \left\{ z' \in X \times Y : \forall z'' \in X \times Y \quad F(z + z'') \geq F(z) + \omega(z', z'') \right\}.$$

In [6, Definition 2.2], the symplectic subdifferential is denoted by XF; here we use different notation.

Definition 2.2 *The symplectic polar, or the symplectic Fenchel transform of $F : X \times Y \to \mathbb{R} \cup \{+\infty\}$, a convex lower semicontinuous function, is the function*

$$F^{*\omega}(z') = \sup \left\{ \omega(z', z) - F(z) : z \in X \times Y \right\}.$$

The particular case $X = Y$. In this case the duality $\langle \cdot, \cdot \rangle$ is a scalar product. Moreover, the space $X \times Y$ is dual with itself, with the duality product

$$\langle\langle (x, y), (x', y') \rangle\rangle = \langle x, x' \rangle + \langle y, y' \rangle.$$

We now introduce the linear function

$$J : X \times Y \to X \times Y, \ J(x, y) = (-y, x)$$

and we remark that we have the relations $J^2 = -I$ and

$$\omega((x, y), (x', y')) = \langle\langle J(x, y), (x', y') \rangle\rangle$$

$$\omega(-Jz', z'') = \langle\langle z', z'' \rangle\rangle.$$

Notice that J makes no sense in the general case when $X \neq Y$.

The subdifferential of $F : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is by definition

$$\partial F(z) = \left\{ z' \in X \times Y : \forall z'' \in X \times Y \quad F(z + z'') \geq F(z) + \langle\langle z', z'' \rangle\rangle \right\}.$$

By comparison with Definition 2.1 of the symplectic subdifferential of F, we obtain

$$z' \in \partial^\omega F(z) \Leftrightarrow Jz' \in \partial F(z)$$

$$z' \in \partial F(z) \Leftrightarrow -Jz' \in \partial^\omega F(z)$$

in the particular case $X = Y$.

We continue by writing the definition of the Fenchel conjugate of a function $F : X \times Y \to \mathbb{R} \cup \{+\infty\}$:

$$F^*(z) = \sup \left\{ \langle\langle z', z \rangle\rangle - F(z') : z' \in X \times Y \right\}.$$

Remark that, from Definition 2.2 of the symplectic polar of F, we get

$$F^{*\omega}(z) = F^*(Jz)$$

in the particular case $X = Y$.

Let us go back to the general setting, when we do not suppose that $X = Y$. (Then, of course, the function J no longer makes sense.) In this generality we have the following symplectic version of the Fenchel inequality.

Theorem 2.3 *Suppose that ϕ is convex and lower semicontinuous. Then for any $z, z' \in X \times Y$ we have*

$$\phi(z) + \phi^{*\omega}(z') \geq \omega(z', z)$$

and the equality is achieved if and only if $z' \in \partial^\omega \phi(z)$.

Remark 2.4 *In the case $X = Y$ this is just a reformulation of the usual Fenchel inequality. Indeed, in this case we may replace $\phi^{*\omega}(z')$ by $\phi^*(Jz')$ and $\omega(z', z)$ by $\langle\langle Jz', z \rangle\rangle$.*

Proof. Taking inspiration from Remark 2.4, we notice that we can reproduce verbatim the proof of the usual Fenchel inequality, even if we place ourselves in the general case.

By Definition 2.2 of the symplectic polar we have that for any $z, z' \in X \times Y$

$$\phi^{*\omega}(z') \geq \omega(z', z) - \phi(z)$$

which gives the symplectic Fenchel inequality. The equality is attained if and only if $\phi(z)$ and $\phi^{*\omega}(z')$ are finite and moreover

$$\phi^{*\omega}(z') = \omega(z', z) - \phi(z) \geq \omega(z', z + z'') - \phi(z + z'')$$

for any $z'' \in X \times Y$. But this is equivalent to the following: $\phi(z)$ and $\phi^{*\omega}(z')$ are finite and moreover

$$\phi(z + z'') \geq \phi(z) + \omega(z', z'')$$

for any $z'' \in X \times Y$, that is, with $z' \in \partial^{\omega}\phi(z)$. □

Remark 2.5 *We may replace the pair X, Y and the associated symplectic form ω by a symplectic manifold (M, ω). In this case ω becomes a field of antisymmetric bilinear forms, that is,*

$$\omega = \omega_z : T_zM \times T_zM \to \mathbb{R}$$

for any $z \in M$. Here we denote by T_zM the tangent space to M at z. We consider then functions

$$F : TM \to \mathbb{R} \cup \{+\infty\}$$

where TM denotes the tangent bundle of M. We require that the function F is lower semicontinuous, and moreover for any $z \in M$ the function $Z \in T_zM \mapsto F(z, Z)$ is convex.

In this more general setting, the definition of the subdifferential of a function F is the set $\partial^{\omega}F(z, Z) \subset T_zM$ given by

$$\partial^{\omega}F(z, Z) = \left\{Z' \in T_zM : \forall Z'' \in T_zM \quad F(z, Z + Z'') \geq F(z, Z) + \omega_z(Z', Z''\right\}$$

and the definition of the symplectic polar changes to the following: for any $z \in M$ and any $Z' \in T_zM$

$$F^{*\omega}(z, Z') = \sup\left\{\omega_z(Z', Z) - F(z, Z) : z \in T_zM\right\}.$$

Finally, the symplectic Fenchel inequality takes the following form: for any $z \in M$ and for any $Z, Z' \in T_zM$ we have

$$\phi(z, Z) + \phi^{*\omega}(z, Z') \geq \omega_z(Z', Z)$$

and the equality is realized if and only if $Z' \in \partial^{\omega}\phi(z, Z)$.

3. Hamiltonian evolution

Definition 3.1 *A function $F : X \times Y \to \mathbb{R} \cup \{+\infty\}$ has a symplectic gradient in a point $z = (x, y)$ if $F(x, y) < +\infty$ and there exists $XF(x, y) = (u, v) \in X \times Y$, called the symplectic gradient of F in (x, y), such that*

(a) *for all $y' \in Y$ we have*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[F(x, y + \varepsilon y') - F(x, y)\right] = \langle u, y' \rangle;$$

(b) *for all $x' \in X$ we have*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[F(x + \varepsilon x', y) - F(x, y)\right] = -\langle x', v \rangle.$$

Let us denote by $DF(x,y) \in X \times Y$ the vector defined by the following: for any $(x',y') \in X \times Y$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[F(x+\varepsilon x', y+\varepsilon y') - F(x,y)\right] = \langle\langle DF(x,y),(x',y')\rangle\rangle.$$

Then, in the particular case $X = Y$, by denoting $z = (x,y)$, we have $JXF(z) = DF(z)$ and $XF(z) = -JDF(z)$. Moreover, we have the following useful formula, obtained from the ones about the symplectic subdifferential in the case $X = Y$:

$$XF(z) \in \partial^{\omega} F(z) \Leftrightarrow DF(z) \in \partial F(z).$$

Moreover, suppose that we use a function $F = F(t,z)$ defined over $\mathbb{R} \times X \times Y$, where we see $X \times Y$ as the symplectic manifold M, as in Remark 2.5. Then, from the formula

$$\omega(XF(t,z), Z') = \langle\langle D_z F(t,z), Z'\rangle\rangle$$

we obtain the following: for any smooth curve $t \mapsto z(t) \in X \times Y$ we denote by $\dot{z}(t)$ the derivative of the curve with respect to t and we have

$$\frac{d}{dt}F(t,z(t)) = D_t F(t,z(t)) + \omega(XF(t,z),\dot{z}(t)). \tag{1}$$

Definition 3.2 *Given a function $H = H(t,x,y) = H(t,z)$ called the Hamiltonian, a curve $z : [0,T] \to X \times Y$ is an evolution curve of that Hamiltonian function if it satisfies the equation*

$$\dot{z}(t) = XH(t,z(t)). \tag{2}$$

In the particular case $X = Y$, let us consider the following example of a Hamiltonian (we use the notation $z = (q,p) \in X \times Y$, with the interpretation that q is a position and p is momentum):

$$H(t,z) = H(t,q,p) = \frac{1}{2m}\langle p,p\rangle^2 + W(q) - \langle f(t),q\rangle.$$

Here m has the interpretation of mass, W is energy and $f(t)$ is an external force. Equation (2) is then

$$(\dot{q},\dot{p}) = (D_p H(t,q,p), -D_q H(t,q,p))$$

which is equivalent to the following system:

$$\begin{cases} \dot{q} = \frac{1}{m}p \\ \dot{p} = -D_q W(q) + f(t) \end{cases}$$

which is the same as

$$m\ddot{q} = -D_q W(q) + f(t).$$

This is the Euler–Lagrange equation for the Lagrangian function

$$L(t,q,\dot{q}) = \frac{1}{2}m\langle\dot{q},\dot{q}\rangle - W(q) + \langle f(t),q\rangle$$

which describes a mechanical system with position q, kinetic energy $\frac{1}{2}m\langle\dot{q},\dot{q}\rangle$, and potential energy $W(q)$, subjected to external forces $f(t)$.

If we stay on the Hamiltonian side, we remark that the system does not dissipate, in the sense that for $z(t) = (q(t),p(t))$ a solution of equation (2), we have

$$0 = \omega(XH(t,z(t)), \ \dot{z}(t)) = \frac{d}{dt}H(t,z(t)) - D_t H(t,z(t)).$$

For example, if $D_t H(t, z(t)) = 0$ (in particular if $H = H(z)$ only) then the Hamiltonian H is constant along any evolution curve.

If we look at the Lagrangian side then we remark that a complete model also needs equations which describe the dissipative behaviour of the system.

We shall later need the following definition of the Poisson bracket.

Definition 3.3 *Let $Der(X, Y)$ be the linear space of functions $f : X \times Y \to \mathbb{R}$ which have a continuous symplectic gradient.*

The Poisson bracket is the bilinear, antisymmetric form

$$\{\cdot, \cdot\} : Der(X, Y) \times Der(X, Y) \to \mathbb{R}^{X \times Y}$$

defined by $\{f, g\} = \omega(Xf, Xg)$.

4. The symplectic BEN principle

In this section we propose a modification of the Hamiltonian formalism with the effect that it can apply to dissipative systems. This proposal builds further on the initial one of Hamiltonian systems with convex dissipation introduced in [6].

A Hamiltonian $H = H(t, x, y) = H(t, z)$ and a dissipation potential $\phi = \phi(\dot{z})$,

$$\phi : X \times Y \to \mathbb{R} \cup \{+\infty\},$$

are given. We shall suppose that the dissipation potential ϕ is convex and lower semicontinuous.

For any evolution curve $z : [0, T] \to X \times Y$, we consider the additive decomposition of $\dot{z}$ into 'reversible' and 'irreversible' parts:

$$\dot{z} = \dot{z}_R + \dot{z}_I, \quad \dot{z}_R = XH(z), \ \dot{z}_I = \dot{z} - XH(z). \tag{3}$$

Alternative good names would be 'conservative' instead of 'reversible' and 'non-conservative' instead of 'irreversible'.

If the evolution is Hamiltonian, with H as the Hamiltonian function, that is, $\dot{z} = XH(z)$, then $\dot{z}_I = 0$ and $\dot{z}_R = \dot{z}$.

We propose the following principle (definition).

Definition 4.1 *(The symplectic BEN principle.) An evolution curve $t \in [0, T] \mapsto z(t) \in X \times Y$ satisfies the symplectic BEN principle for the Hamiltonian H and dissipation potential ϕ if for almost any $t \in [0, T]$ we have*

$$\phi(\dot{z}(t)) + \phi^{*\omega}(\dot{z}_I(t)) = \omega(\dot{z}_I(t), \dot{z}(t)). \tag{4}$$

Remark 4.2 *We use the name 'BEN principle' because, as explained in Section 5, relation (24), in the particular case of standard plasticity the symplectic BEN principle reduces to the BEN principle [3–5] if we neglect the dynamical terms.*

Two other equivalent forms of the symplectic BEN principle are given in the next proposition.

Proposition 4.3 *An evolution curve $t \in [0, T] \mapsto z(t) \in X \times Y$ satisfies the symplectic BEN principle for the Hamiltonian H and dissipation potential ϕ if and only if it satisfies one of the following.*

(a) For almost every $t \in [0, T]$

$$\dot{z}(t) - XH(t, z(t)) \in \partial^{\omega}\phi(\dot{z}(t)). \tag{5}$$

(b) The evolution curve minimizes the functional

$$\Pi(z') = \int_0^T \left\{ \phi(\dot{z'}(t)) + \phi^{*\omega}(\dot{z}'_I(t)) - \frac{\partial H}{\partial t}(t, z'(t)) \right\} dt + H(T, z'(T)) \tag{6}$$

among all curves $z' : [0, T] \to X \times Y$ such that $z'(0) = z(0)$.

Proof. (a) We apply Theorem 2.3 to (4) and we obtain that

$$\dot{z}_I(t) \in \partial^{\omega}\phi(\dot{z}(t)).$$

But this is the same as (5), by using the decomposition (3).

(b) According to the symplectic Fenchel inequality (Theorem 2.3), we know that for any evolution curve z'

$$\int_0^T \left\{\phi(\dot{z'}(t)) + \phi^{*\omega}(\dot{z'}_I(t)) - \omega(\dot{z'}_I(t), \dot{z'}(t))\right\} \, \mathrm{dt} \geq 0 \tag{7}$$

and that the equality is realized by a curve z which satisfies the symplectic BEN principle. Besides, taking into account the antisymmetry of ω and (3), we have

$$\begin{aligned} -\omega(\dot{z'}_I(t), \dot{z'}(t)) &= -\omega(\dot{z}'(t) - XH(z'(t)), \dot{z}'(t)) \\ &= \omega(XH(z'(t)), \dot{z}'(t)) = \frac{d}{dt}(H \circ z')(t) \\ &= \frac{d}{dt}(H(t, z'(t))) - \frac{\partial H}{\partial t}(t, z'(t)). \end{aligned}$$

Introducing the previous expression into (7), and taking into account the initial condition and (6), we have

$$\int_0^T \left\{\phi(\dot{z'}(t)) + \phi^{*\omega}(\dot{z'}_I(t)) - \omega(\dot{z'}_I(t), \dot{z'}_R(t))\right\} \, \mathrm{dt} = \Pi(z') - H(0, z'(0)) \geq 0,$$

and therefore any solution of (4) which satisfies the initial condition is also a solution of (6). Conversely, any solution of (6) satisfies the initial condition and it satisfies (4) for almost every $t \in [0, T]$. □

Remark 4.4 *The symplectic BEN principle in the form (5) appears in [6, Definition 2.3, relation (14)]. The notation is slightly different: in the mentioned reference a 'dissipation function' $\mathcal{R}$ is used, which corresponds to the dissipation potential ϕ here. Notice however that in this article we do not take as a hypothesis that ϕ is a non-negative function, like in [6, Definition 2.3(a)]. With the formulation proposed here, there is a different possible positivity hypothesis which we might add. Indeed, because of the antisymmetry of the symplectic form, when we use relation (1) for the function H we get*

$$\omega(\dot{z}_I, \dot{z}) = -\omega(XH, \dot{z}) = -\frac{d}{dt}\left[H(t, z(t))\right] + \frac{\partial H}{\partial t}(t, z(t)).$$

From the symplectic BEN principle (8) we obtain the following energy balance:

$$\phi(z, \dot{z}) + (\phi\,(z, \cdot))^{*\omega}\,(\dot{z}_I) \;=\; -\frac{d}{dt}\left[H(t, z(t))\right] + \frac{\partial H}{\partial t}(t, z(t)).$$

If the system dissipates then we expect that the quantity from the right-hand side of the previous equation is non-negative. A sufficient condition is that ϕ satisfies the following: for any $z, z' \in X \times Y$,

$$\phi(z) + \phi^{*\omega}(z') \geq 0.$$

In the particular case when ϕ is positively 1-homogeneous (as happens for many interesting physical systems), $\phi^{*\omega}$ takes only the values $0, +\infty$. Then, for positively 1-homogeneous dissipation ϕ we have:

$$\textit{if}\ \phi^{*\omega}(z') < +\infty \quad \textit{then} \quad \phi(z) + \phi^{*\omega}(z') \;=\; \phi(z).$$

Therefore in this case the positivity of $\phi(z)$ is equivalent to the positivity of $\phi(z) + \phi^{*\omega}(z')$.

We may generalize, in the setting explained in Remark 2.5, the three variants of the symplectic BEN principle by making $\phi = \phi(z, \dot{z})$ and writing for example instead of (4) the following:

$$\phi(z, \dot{z}) + (\phi\,(z, \cdot))^{*\omega}\,(\dot{z}_I) = \omega(\dot{z}_I, \dot{z}_R). \tag{8}$$

Invariance of the symplectic BEN principle. The symplectic BEN principle (8) or its equivalent formulation (5) are stable under reparameterization. This means the following. Mathematically the dissipation potential ϕ is a function defined on the tangent bundle of $X \times Y$. Consider then a reparameterization $(x', y') = \Psi(x, y)$ which preserves the symplectic form, in the sense that the function $\Psi : X \times Y \to X \times Y$ is bijective and differentiable everywhere, and that if we denote by $D\Psi(x, y) \in Lin(X \times Y, X \times Y)$ its derivative at (x, y) then for any pair of 'tangent vectors' $(x_1, y_1), (x_2, y_2) \in X \times Y$ we have

$$\omega(D\Psi(x,y)(x_1,y_1), D\Psi(x,y)(x_2,y_2)) = \omega((x_1,y_1),(x_2,y_2)).$$

Then, in the new coordinates (x', y') the dissipation potential transforms as

$$\phi'(\Psi(x,y), D\Psi(x,y)(x_1,y_1)) = \phi((x,y),(x_1,y_1)).$$

The Hamiltonian H becomes $H'(t, \Psi(x,y)) = H(t,x,y))$.

The symplectic polar of $\phi((x,y),\cdot)$ will be, in the new coordinates,

$$\begin{aligned}
&\left(\phi'\left(\Psi(x,y),\cdot\right)\right)^{*\omega}\left(D\Psi(x,y)(x_1,y_1)\right)\\
&= \sup\left\{\omega(D\Psi(x,y)(x_1,y_1),(x',y')) - \phi(\Psi(x,y),(x',y')) : (x',y') \in X \times Y\right\}\\
&= \sup\{\omega(D\Psi(x,y)(x_1,y_1), D\Psi(x,y)(x_2,y_2))\\
&\quad - \phi'(\Psi(x,y), D\Psi(x,y)(x_2,y_2)) : (x_2,y_2) \in X \times Y\}\\
&= \sup\{\omega((x_1,y_1),(x_2,y_2) - \phi((x,y),(x_2,y_2)) : (x_2,y_2) \in X \times Y\}\\
&= (\phi\left((x,y),\cdot\right))^{*\omega}\left((x_1,y_1)\right).
\end{aligned}$$

In the new coordinates the curve $z(t) = (x(t), y(t))$ becomes $z'(t) = \Psi(x(t), y(t))$, with the associated velocity

$$\dot{z}'(t) = D\Psi(x(t),y(t))\dot{z}(t).$$

Therefore the symplectic BEN principle (8) expressed in the new coordinates, with ϕ' and H' instead of ϕ and H, will be the same as the original one.

Let us explore further the consequences of the symplectic BEN principle.

Proposition 4.5 *The curve $z : [0,T] \to X \times Y$ is a solution of the symplectic BEN principle (8) if and only if for almost every $t \in [0,T]$ and for any function $f : [0,T] \to Der(X \times Y)$, $f = f(t,z)$ which is derivable with respect to t we have*

$$\begin{aligned}
\phi(z(t),\dot{z}(t) - Xf(t,z(t))) &\geq \frac{d}{dt}\left[f(t,z(t))\right] - \frac{\partial f}{\partial t}(t,z(t)) \\
&\quad + \omega(XH(t,z(t)), Xf(t,z(t))) + \phi(z(t),\dot{z}(t)).
\end{aligned} \tag{9}$$

Moreover, if the curve z satisfies (9) for any $f \in Der(X \times Y)$ (i.e. f does not depend on time) then z is a solution of the symplectic BEN principle (8).

Proof. We know that (8) is equivalent to (5), and more precisely we know that z satisfies (8) if and only if it satisfies the following:

$$\dot{z} - XH(t,z(t)) \in \partial^{\omega}\phi(z(t),\cdot)(\dot{z}). \tag{10}$$

The relation (10) means that for any $t \in [0,T]$ and for any $z' \in X \times Y$ we have

$$\phi(z(t),\dot{z}(t) + z') \geq \omega(\dot{z}(t) - XH(t,z(t)), z') + \phi(z(t),\dot{z}(t)). \tag{11}$$

Let us now choose $z' = -Xf(t,z(t))$. We then get

$$\begin{aligned}
\phi(z(t),\dot{z}(t) - Xf(t,z(t))) &\geq \omega(Xf(t,z(t)),\dot{z}(t)) \\
&\quad + \omega(XH(t,z(t)), Xf(t,z(t))) + \phi(z(t),\dot{z}(t)).
\end{aligned} \tag{12}$$

But this is the same as (9) because

$$\omega(Xf(t,z(t)),\dot{z}(t)) \;=\; \frac{d}{dt}\,[f(t,z(t))] - \frac{\partial f}{\partial t}(t,z(t))$$

by the definition of the symplectic gradient.

For the converse implication, for any $z' \in X \times Y$ let us pick the function $f(z) = \omega(z,z')$. Then $Xf(z) = -z'$ for any $z \in X \times Y$, and therefore we can trace back our steps from (9) applied for this choice of f to (11). Because this can be done for any $z' \in X \times Y$ it follows that (11) is true for any $z' \in X \times Y$, which proves that the curve z satisfies the symplectic BEN principle (10). □

Proposition 4.6 *Suppose that the curve $z : [0,T] \to X \times Y$ is a solution of the symplectic BEN principle (8). Then for almost every $t \in [0,T]$ we have*

$$\phi(z(t),\dot{z}(t) - Xf(t,z(t))) \;\geq\; \omega(Xf(t,z(t)),\dot{z}(t)) + \phi(z(t),\dot{z}(t)) \tag{13}$$

for any function f with the property that

$$\omega(Xf(t,z(t)),XH(t,z(t))) = 0,$$

in other words, for any integral of motion of the Hamiltonian H.

Proof. The hypothesis implies that z and f satisfy (12). We then use the fact that f is an integral of motion of the Hamiltonian H in order to get (13). □

Proposition 4.7 *Suppose that the curve $z : [0,T] \to X \times Y$ is a solution of the symplectic BEN principle (8). Then for any function $f : [0,T] \to Der(X \times Y)$, $f = f(t,x,y)$ which is derivable with respect to t we have*

$$\int_0^T [\phi(z(t),\dot{z}(t) - Xf(t,z(t))) - \phi(z(t),\dot{z}(t)]\,dt \tag{14}$$

$$\geq f(T,z(T)) - f(0,z(0)) \;+\; \int_0^T \left[\{H,f\}\,(t,z(t)) - \frac{\partial f}{\partial t}(t,z(t))\right] dt.$$

Conversely, any solution of the problem (14) is also a solution of the symplectic BEN principle (8) for almost every $t \in [0,T]$.

Proof. By integration with respect to time of (9), and then by integration by parts, we obtain (14). Conversely, we may pick the 'test functions' f to be with compact support with respect to the time variable, which gives the second claim. □

Proposition 4.8 *Any solution of the problem (14) satisfies the dissipation balance equation: for every $\tau \in [0,T]$,*

$$\int_0^\tau \left[\phi(z(t),\dot{z}(t)) + (\phi(z(t),\cdot))^{*\omega}\,(\dot{z}(t) - XH(t,z(t)))\right] dt \tag{15}$$

$$= H(0,z(0)) - H(\tau,z(\tau)) \;+\; \int_0^\tau \frac{\partial H}{\partial t}(t,z(t))dt$$

and the dissipation inequality

$$\phi(z(t),\dot{z}_I(t)) \;\geq\; \frac{d}{dt}\,[H(t,z(t))] - \frac{\partial H}{\partial t}(t,z(t)) + \phi(z(t),\dot{z}(t)). \tag{16}$$

Proof. Indeed, solutions of (14) satisfy the symplectic BEN principle (8) for almost every $t \in [0, T]$. We integrate it from 0 to τ and we obtain the dissipation balance equation. For the inequality (16) we use Proposition 4.6 for $f = H$. □

It is interesting to note that the dissipation balance equation (15) is a generalization of [6, Theorem 2.7, relation (21)]. The mentioned theorem has among its hypotheses that the dissipation potential is positively 1-homogeneous:

$$\forall \lambda \geq 0, \qquad \phi(z, \lambda \dot{z}) = \lambda\, \phi(z, \dot{z}).$$

Here this hypothesis is not required. The relation (15) is a generalization of [6, relation (21)] because in the case where the dissipation ϕ is positively 1-homogeneous then, as explained in Remark 4.4, we have $\phi(z, \dot{z}) + \phi^{*\omega}(z, \dot{z}') = \phi(z, \dot{z})$ whenever the quantity on the left-hand side is finite.

5. Application: Standard plasticity

We would now like to illustrate the general formalism and to show how it allows us to develop powerful variational principles for dissipative systems within the frame of continuum mechanics. To begin with, we tackle the standard plasticity in small deformations based on the additive decomposition of strains into reversible and irreversible strains:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_R + \boldsymbol{\varepsilon}_I$$

where $\boldsymbol{\varepsilon}_I$ is a plastic strain $\boldsymbol{\varepsilon}_p$.

The present modelling can immediately be extended to similar constitutive laws by alternatively considering viscous or viscoplastic strains depending on the material behaviour.

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, with piecewise smooth boundary $\partial\Omega$. As usual, it is divided into two disjoint parts, $\partial\Omega_0$ (called support) where the displacements are imposed, and $\partial\Omega_1$ where the surface forces are imposed. The elements of the space X are fields $x = (\boldsymbol{u}, \boldsymbol{\varepsilon}_I) \in U \times E$ where $\boldsymbol{\varepsilon}_I$ is the irreversible strain field and $\boldsymbol{u}$ is a displacement field on the body Ω with trace $\bar{\boldsymbol{u}}$ on $\partial\Omega$. The elements of the corresponding dual space Y are of the form $y = (\boldsymbol{p}, \boldsymbol{\pi})$. Unlike $\boldsymbol{p}$ which is clearly the linear momentum, we do not know at this stage the physical meaning of $\boldsymbol{\pi}$. We let $z = (x, y)$.

The duality between the spaces X and Y has the form

$$\langle x, y \rangle = \int_\Omega \left(\langle \boldsymbol{u}, \boldsymbol{p} \rangle + \langle \boldsymbol{\varepsilon}_I, \boldsymbol{\pi} \rangle \right)\, \mathrm{d}x$$

where the duality products which appear in the integral are finite-dimensional duality products on the image of the fields $\boldsymbol{u}, \boldsymbol{p}$ (for our example this means a scalar product on $\mathbb{R}^3$) and on the image of the fields $\boldsymbol{\varepsilon}, \boldsymbol{\pi}$ (in this case this is a scalar product on the space of 3×3 symmetric matrices). We denote all these standard dualities by the same $\langle \cdot, \cdot \rangle$ symbols.

The total Hamiltonian of the structure is taken in the integral form

$$H(t, z) = \int_\Omega \left\{ \frac{1}{2\rho} \parallel \boldsymbol{p} \parallel^2 + w(\nabla \boldsymbol{u} - \boldsymbol{\varepsilon}_I) - \boldsymbol{f}(t) \cdot \boldsymbol{u} \right\} - \int_{\partial\Omega_1} \bar{\boldsymbol{f}}(t) \cdot \boldsymbol{u}.$$

The first term is the kinetic energy, w is the elastic strain energy, $\boldsymbol{f}$ is the volume force and $\bar{\boldsymbol{f}}$ is the surface force on the part $\partial\Omega_1$ of the boundary, the displacement field being equal to an imposed value $\bar{\boldsymbol{u}}$ on the remaining part $\partial\Omega_0$.

Its symplectic gradient, according to Definition 3.1, is

$$XH = ((D_{\boldsymbol{p}} H, D_{\boldsymbol{\pi}} H), (-D_{\boldsymbol{u}} H, -D_{\boldsymbol{\varepsilon}_I} H))$$

where, introducing as usual the stress field

$$\boldsymbol{\sigma} = Dw(\nabla \boldsymbol{u} - \boldsymbol{\varepsilon}_I),$$

$D_{\boldsymbol{u}} H$ is the gradient in the variational sense (from Definition 3.1 and the integral form of the duality product)

$$D_{\boldsymbol{u}} H = \frac{\partial H}{\partial \boldsymbol{u}} - \nabla \cdot \left(\frac{\partial H}{\partial \nabla \boldsymbol{u}} \right) = -\boldsymbol{f} - \nabla \cdot \boldsymbol{\sigma}$$

and

$$D_{\bar{u}} H = \boldsymbol{\sigma} \cdot \boldsymbol{n} - \bar{\boldsymbol{f}}.$$

Thus we have

$$\dot{z}_I = \dot{z} - XH = \left(\left(\dot{\boldsymbol{u}} - \frac{\boldsymbol{p}}{\rho}, \dot{\boldsymbol{\varepsilon}}_I \right), (\dot{\boldsymbol{p}} - \boldsymbol{f} - \nabla \cdot \boldsymbol{\sigma}, \dot{\boldsymbol{\pi}} - \boldsymbol{\sigma}) \right).$$

We shall use a dissipation potential which has an integral form:

$$\Phi(z) = \int_\Omega \phi(\boldsymbol{p}, \boldsymbol{\pi})\, \mathrm{d}x$$

and we shall assume that the symplectic Fenchel transform of Φ is expressed as the integral of the symplectic Fenchel transform of the dissipation potential density ϕ.

The symplectic Fenchel transform of the function ϕ reads

$$\phi^{*\omega}(\dot{z}_I) = \sup \left\{ \langle \dot{\boldsymbol{u}}_I, \dot{\boldsymbol{p}}' \rangle + \langle \dot{\boldsymbol{\varepsilon}}_I, \dot{\boldsymbol{\pi}}' \rangle - \langle \dot{\boldsymbol{u}}', \dot{\boldsymbol{p}}_I \rangle - \langle \dot{\boldsymbol{\varepsilon}}_I', \dot{\boldsymbol{p}}_I \rangle - \phi(\dot{z}') : \dot{z}' \in X \times Y \right\}.$$

To recover the standard plasticity, we suppose that ϕ depends explicitly only on $\dot{\boldsymbol{\pi}}$:

$$\phi(\dot{z}) = \varphi(\dot{\boldsymbol{\pi}}).$$

Denoting by χ_K the indicator function of a set K (equal to 0 on K and to $+\infty$ otherwise), we obtain

$$\phi^{*\omega}(\dot{z}_I) = \chi_{\{\mathbf{0}\}}(\dot{\boldsymbol{u}}_I) + \chi_{\{\mathbf{0}\}}(\dot{\boldsymbol{p}}_I) + \chi_{\{\mathbf{0}\}}(\dot{\boldsymbol{\pi}}_I) + \varphi^*(\dot{\boldsymbol{\varepsilon}}_I)$$

where φ^* is the usual Fenchel transform. In other words, the quantity $\phi^{*\omega}(\dot{z}_I)$ is finite if and only if all of the following are true:

(a) $\phi^{*\omega}(\dot{z}_I) = \varphi^*(\dot{\boldsymbol{\varepsilon}}_I)$;

(b) $\boldsymbol{p}$ equals the linear momentum:

$$\boldsymbol{p} = \rho \dot{\boldsymbol{u}}; \tag{17}$$

(c) the balance of linear momentum is satisfied:

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \dot{\boldsymbol{p}} = \rho \ddot{\boldsymbol{u}} \quad \text{on} \quad \Omega, \qquad \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{f}} \quad \text{on} \quad \partial\Omega_1; \tag{18}$$

(d) an equality which reveals the meaning of the variable π:

$$\dot{\boldsymbol{\pi}} = \boldsymbol{\sigma}. \tag{19}$$

The symplectic BEN principle applied to standard plasticity states that the evolution curve minimizes

$$\Pi(z) = \int_0^T \left\{ \varphi(\dot{\boldsymbol{\pi}}) + \varphi^*(\dot{\boldsymbol{\varepsilon}}_I) - \frac{\partial H}{\partial t}(t, z) \right\} \mathrm{d}t + H(T, z(T)) \tag{20}$$

among all curves $z : [0, T] \to X \times Y$ such that $z(0) = (x_0, y_0)$, the kinematical conditions on $\partial\Omega_0$, (17), (18) and (19) are satisfied.

The symplectic BEN principle and the original BEN principle. Let us examine the important case where the kinetic energy and inertia forces can be neglected (quasi-static behaviour):

$$\dot{\boldsymbol{p}} = \mathbf{0}, \qquad H(t, z) = \int_\Omega \{ w(\nabla \boldsymbol{u} - \boldsymbol{\varepsilon}_I) - \boldsymbol{f}(t) \cdot \boldsymbol{u} \} - \int_{\partial\Omega_1} \bar{\boldsymbol{f}}(t) \cdot \boldsymbol{u}$$

and the elasticity is linear:

$$\dot{\boldsymbol{\varepsilon}}_I = \nabla \dot{\boldsymbol{u}} - \boldsymbol{S} \dot{\boldsymbol{\sigma}}$$

denoting $\boldsymbol{S} = (Dw)^{-1}$ the compliance operator. Eliminating $\boldsymbol{\pi}$ and $\boldsymbol{p}$ thanks to (17) and (19), the symplectic BEN principle (20) is transformed and claims that for a given curve $\boldsymbol{u} : [0, T] \to U$ which satisfies the kinematical conditions, the evolution curve minimizes

$$\Pi(\boldsymbol{\sigma}) = \int_0^T \left\{ \varphi(\boldsymbol{\sigma}) + \varphi^*(\nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}}) - \frac{\partial H}{\partial t}(t, z) \right\} \, \mathrm{d}t + H(T, z(T)) \tag{21}$$

among all curves $\boldsymbol{\sigma} : [0, T] \to E$ such that $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ and

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \quad \text{on} \quad \Omega, \qquad \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{f}} \quad \text{on} \quad \partial\Omega_1 \tag{22}$$

are satisfied. This expression can be transformed as follows. For the sake of ease, let

$$\langle \boldsymbol{l}(t), u \rangle = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{u} + \int_{\partial\Omega_1} \bar{\boldsymbol{f}}(t) \cdot \boldsymbol{u}.$$

Then,

$$\frac{\partial H}{\partial t}(t, z) = -\langle \dot{\boldsymbol{l}}(t), \boldsymbol{u} \rangle.$$

On the other hand

$$\frac{d}{dt}[H(t, z(t))] = \langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} - \dot{\boldsymbol{\varepsilon}}_I \rangle - \langle \boldsymbol{l}(t), \dot{\boldsymbol{u}} \rangle - \langle \dot{\boldsymbol{l}}(t), u \rangle.$$

For the minimizer, the kinematical conditions on $\partial\Omega_0$ and the equilibrium equations (22) are satisfied, and using Green's formula

$$\langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} \rangle = \langle \boldsymbol{l}(t), \dot{\boldsymbol{u}} \rangle, \tag{23}$$

which leads to

$$\frac{d}{dt}[H(t, z(t))] - \frac{\partial H}{\partial t}(t, z) = -\langle \boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}_I \rangle.$$

Time-integrating and replacing in (21) gives

$$\Pi(\boldsymbol{\sigma}) = \int_0^T \left\{ \varphi(\boldsymbol{\sigma}) + \varphi^*(\nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}}) - \langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}} \rangle \right\} \, \mathrm{d}t + H(0, z(0)).$$

Integrating the latter term of the integral and forgetting the constant leads to the original BEN principle [3–5]. The evolution curve minimizes

$$\bar{\Pi}(\boldsymbol{\sigma}) = \int_0^T \left\{ \varphi(\boldsymbol{\sigma}) + \varphi^*(\nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}}) - \langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} \rangle \right\} \, \mathrm{d}t + \frac{1}{2} \langle \boldsymbol{\sigma}(T), \boldsymbol{S}\boldsymbol{\sigma}(T) \rangle \tag{24}$$

among all curves $\boldsymbol{\sigma} : [0, T] \to E$ such that $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ and equilibrium equations (22) are satisfied.

6. Concluding remarks and future work

It is worth remarking that in general the displacement evolution is also an unknown of the structure problem. This suggests considering another variant formulation relaxing the equilibrium equations. Once again using Green's formula (23) for the minimizer leads to the claim that the evolution curve minimizes

$$\bar{\Pi}(\boldsymbol{u}, \boldsymbol{\sigma}) = \int_0^T \left\{ \varphi(\boldsymbol{\sigma}) + \varphi^*(\nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}}) - \langle \boldsymbol{l}(t), \dot{\boldsymbol{u}} \rangle \right\} \, \mathrm{d}t + \frac{1}{2} \langle \boldsymbol{\sigma}(T), \boldsymbol{S}\boldsymbol{\sigma}(T) \rangle \tag{25}$$

among all curves $(\boldsymbol{u}, \boldsymbol{\sigma}) : [0, T] \to U \times E$ such that $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ and the kinematical conditions on $\partial\Omega_0$ are satisfied. The variation with respect to the stress field allows us to recover the dissipative constitutive law while the one with respect to the velocity field allows us to recover the equilibrium equations. This space-time variational principle turns out to be a powerful alternative to the classical step-by-step approaches in the sense that it works on the whole evolution of the system in the spirit of Ladevèze's Large Time INcrement (LATIN) method [43–47]. It averts typical pitfalls of step-by-step approaches which cumulate errors step after step and may fail in the case of non-convergence at a given step.

In the future, we wish to develop the applications of this new theoretical formalism according to the three objectives mentioned in the introduction.

- Extending to the dissipative systems the geometrical methods of classical dynamics by a suitable change of coordinates (in particular using canonical transformations, Lie group theory and momentum maps).
- Exploring the dissipative rheological models in dynamical situations and using the dynamical BEN principle to solve evolution problems using suitable numerical algorithms.
- We also aim to generalize this approach to implicit standard materials by introducing a symplectic bipotential.

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