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Lower semi-continuity of integrals with G-quasiconvex potential

Marius Buliga

Abstract. This paper introduces the proper notion of variational quasiconvexity associated to a group of bi-Lipschitz homeomorphisms, with possible applications in elasticity. We prove a lower semicontinuity theorem connected to this notion, which improves a result of Dacorogna and Fusco [7]. In the second part of the paper we apply this result to a class of functions, introduced in [5]. Such functions are $\mathrm{GL}(n, \mathrm{R})^+$ quasiconvex, hence they induce lower semicontinuous integrals.

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1. Introduction

Lower semi-continuity of variational integrals

$$u \ \mapsto \ \mathrm{I}(u) = \int_\Omega w(\mathrm{D}u(x)) \ \mathrm{d}x$$

defined over Sobolev spaces is connected to the convexity of the potential w. In the scalar case, that is for functions u with domain or range in R, the functional I is weakly $\mathrm{W}^{1,p}$ lower semi-continuous (weakly * $\mathrm{W}^{1,\infty}$) if and only if w is convex, provided it is continuous and satisfies some growth conditions. The notion which replaces convexity in the vector case is quasiconvexity (introduced by Morrey [14]).

We shall concentrate on the case $u : \Omega \subset \mathrm{R}^n \to \mathrm{R}^n$ which is interesting for continuum media mechanics. Standard notation will be used, like:

$gl(n, \mathrm{R})$	the linear space (Lie algebra) of $n \times n$ real matrices
$\mathrm{GL}(n, \mathrm{R})$	the group of invertible $n \times n$ real matrices
$\mathrm{GL}(n, \mathrm{R})^+$	the group of matrices with positive determinant
$sl(n, \mathrm{R})$	the algebra of traceless $n \times n$ real matrices
$\mathrm{SL}(n, \mathrm{R})$	the group of real matrices with determinant one

$\mathrm{CO}(n)$	the group of conformal matrices
id	the identity map
1	the identity matrix
$\circ$	function composition

In this frame Morrey's quasiconvexity has the following definition.

Definition 1.1. *Let $\Omega \subset \mathrm{R}^n$ be an open bounded set such that $\mid \partial\Omega \mid = 0$ and $w : gl(n, \mathrm{R}) \rightarrow \mathrm{R}$ be a measurable function. The map w is quasiconvex if for any $H \in gl(n, \mathrm{R})$ and any Lipschitz $\eta : \Omega \rightarrow \mathrm{R}^n$, such that $\eta(x) = 0$ on $\partial\Omega$, we have*

$$\int_{\Omega} w(H) \ \leq \ \int_{\Omega} w(H + \mathrm{D}\eta(x)) \tag{1}$$

Translation and rescaling arguments show that the choice of Ω is irrelevant in the above definition.

In elasticity the elastic potential function w is not defined on the Lie algebra $gl(n, \mathrm{R})$ but on the Lie group $\mathrm{GL}(n, \mathrm{R})$ or a subgroup of it. It would be therefore interesting to find the connections between lower semicontinuity of the functional and the (well chosen notion of) quasiconvexity in this non-linear context. This is a problem which floats in the air for a long time. Let us recall two different definitions of quasiconvexity which are relevant.

Definition 1.2. *Let $w : \mathrm{GL}(n, \mathrm{R})^+ \rightarrow \mathrm{R}$. Then:*

(a) *(Ball [2]) w is quasiconvex if for any $\mathrm{F} \in \mathrm{GL}(n, \mathrm{R})^+$ and any $\eta \in \mathrm{C}_c^\infty(\Omega, \mathrm{R}^n)$ such that $\mathrm{F} + \mathrm{D}\eta(x) \in \mathrm{GL}(n, \mathrm{R})^+$ for almost any $x \in \Omega$ we have*

$$\int_{\Omega} w(\mathrm{F} + \mathrm{D}\eta(x)) \ \geq \mid \Omega \mid w(\mathrm{F})$$

(b) *(Giaquinta, Modica & Soucek [10], page 174, definition 3) w is Diff-quasiconvex if for any diffeomorphism $\phi : \Omega \rightarrow \phi(\Omega)$ such that $\phi(x) = \mathrm{F}x$ on $\partial\Omega$, for some $\mathrm{F} \in \mathrm{GL}(n, \mathrm{R})^+$ we have:*

$$\int_{\Omega} w(\mathrm{D}\phi(x)) \ \geq \ \int_{\Omega} w(\mathrm{F})$$

These two definitions are equivalent.

It turns out that very little is known about the lower semicontinuity properties of integrals given by Diff-quasiconvex potentials. It is straightforward that Diff-quasiconvexity is a necessary condition for weakly * $\mathrm{W}^{1,\infty}$ (or uniform convergence of Lipschitz mappings) (see [10] Proposition 2, same page).

More is known about the properties of polyconvex maps. A polyconvex map $w : \mathrm{GL}(n, \mathrm{R})^+ \rightarrow \mathrm{R}$ is described by a convex function $g : \mathrm{D} \subset \mathrm{R}^{\mathrm{M}} \rightarrow \mathrm{R}$ (the domain of definition D is convex as well) and M rank one affine functions $\nu_1, ..., \nu_{\mathrm{M}}$: $\mathrm{GL}(n, \mathrm{R})^+ \to \mathrm{R}$ such that for any $\mathrm{F} \in \mathrm{GL}(n, \mathrm{R})^+$

$$w(\mathrm{F}) = g(\nu_1(\mathrm{F}), ..., \nu_{\mathrm{M}}(\mathrm{F}))$$

The rank one affine functions are known(cf. Edelen [8], Ericksen [9], Ball, Curie, Olver [4]): ν is rank one affine if and only if $\nu(\mathrm{F})$ can be expressed as a linear combination of subdeterminants of F. Any rank one convex function is also called a null Lagrangian, because it generates a trivial Euler-Lagrange equation.

Polyconvex function give lower semicontinuous functionals, as a consequence of Jensen's inequality and continuity of (integrals of) null lagrangians. This is a very interesting path to follow (cf. Ball [3]) and it leads to many applications. But it leaves unsolved the problem: are the integrals given by Diff-quasiconvex potentials lower semicontinuous?

In the case of incompressible elasticity one has to work with the group of matrices with determinant one, i.e. $\mathrm{SL}(n, \mathrm{R})$. The "linear" way of thinking has been compensated by wonders of analytical ingenuity. One purpose of this paper is to show how a slight modification of thinking, from linear to nonlinear, may give interesting results in the case $w : \mathrm{G} \to \mathrm{R}$ where G is a Lie subgroup of $\mathrm{GL}(n, \mathrm{R})$. Note that when n is even a group which deserves attention is $\mathrm{S}p(n, \mathrm{R})$, the group of symplectic matrices.

From now on linear transformations of R^n and their matrices are identified. G is a Lie subgroup of $\mathrm{GL}(n, \mathrm{R})$.

Definition 1.3. *For any $\Omega \subset \mathrm{R}^n$ open, bounded, with smooth boundary, we introduce the set $[\mathrm{G}](\Omega)$ of all bi-Lipschitz mappings u from Ω to R^n such that for almost any $x \in \Omega$ we have $\mathrm{D}u(x) \in \mathrm{G}$. The subset $[\mathrm{G}]_c(\Omega)$ contains all $\phi \in [\mathrm{G}](\Omega]$ such that $\phi -$ id has compact support in $\bar{\Omega}$.*

The set $Q \subset \mathrm{R}^n$ is the unit cube $(0, 1)^n$.

The departure point of the paper is the following natural definition.

Definition 1.4. *The continuous function $w : \mathrm{G} \to \mathrm{R}$ is G-quasiconvex if for any $\mathrm{F} \in \mathrm{G}$ and $u \in [\mathrm{G}](Q)$ we have:*

$$\int_Q w(\mathrm{F}) \ dx \ \leq \ \int_Q w(\mathrm{F}\mathrm{D}u(x)) \ dx \qquad (2)$$

We describe now the structure of the paper. After the formulation of the lower semicontinuity Theorem 2.1, in section 3 is shown that quasiconvexity in the sense of definition 1.2 is the same as $\mathrm{GL}(r, n)^+$ quasiconvexity. Theorem 2.1 is proved in section 4. In section 5 is described a class of $\mathrm{GL}(n, \mathrm{R})^+$ quasiconvex functions introduced in Buliga [5]. Theorem 2.1 is used to prove that any such function induces a lower semicontinuous integral.

2. G-quasiconvexity and the lower semicontinuity result

We denote by $[\mathrm{G}]_c$ the class of all Lipschitz mapping from R^n to R^n such that $u - id$ has compact support and for almost any $x \in \mathrm{R}^n$ we have $\mathrm{D}u(x) \in \mathrm{G}$. The main result of the paper is:

Theorem 2.1. *Let* G *be a Lie subgroup of* $\mathrm{GL}(n, \mathrm{R})$, Ω *an open, bounded set with* $\mid \partial\Omega \mid = 0$ *and* $w : \mathrm{G} \to \mathrm{R}$ *locally Lipschitz.*

a) *Suppose that for any sequence* $u_h \in [\mathrm{G}]_c$ *weakly ** $\mathrm{W}^{1,\infty}$ *convergent to* id *we have:*

$$\int_\Omega w(\mathrm{F}) \ dx \ \leq \ \liminf_{h\to\infty} \int_\Omega w(\mathrm{FD}u_h(x)) \ dx \tag{3}$$

Then for any bi-Lipschitz $u \in [\mathrm{G}]_c$ *and for any sequence* u_h *weakly ** $\mathrm{W}^{1,\infty}$ *convergent to* u *we have:*

$$\int_\Omega w(\mathrm{D}u(x)) \ dx \ \leq \ \liminf_{h\to\infty} \int_\Omega w(\mathrm{D}u_h(x)) \ dx \tag{4}$$

Moreover, if (4) *holds for any bi-Lipschitz* $u \in [\mathrm{G}]_c$ *and for any sequence* u_h *weakly ** $\mathrm{W}^{1,\infty}$ *convergent to* u *then* w *is* G*-quasiconvex.*

b) *Suppose that* G *contains the group* $\mathrm{CO}(\mathrm{R}^n)$ *of conformal matrices. Then* (4) *holds for any bi-Lipschitz* $u \in [\mathrm{G}]_c$ *and for any sequence* u_h *weakly ** $\mathrm{W}^{1,\infty}$ *convergent to* u *if and only if* w *is* G*-quasiconvex.*

The fact that weakly * lower semicontinuity implies G quasiconvexity (end of point (a)) is easy to prove by rescaling arguments (cf. Proposition 2, Giaquinta, Modica and Soucek *op. cit.*).

The method of proving the point (a) of the theorem is well known (see Meyers [13]). For the point (b) we have to use a convex integration result of Dacorogna, Marcellini [6], as a replacement of a controlled Lipschitz extension argument not known to be true for bi-Lipschitz maps. In the paper Dacorogna, Fusco [7] the authors needed the hypothesis of "slow homotopies" in order to prove a result which resembles to Theorem 2.1, for the whole group of bi-Lipschitz homeomorphisms.

3. G-quasiconvexity

This section contains preliminary properties of G-quasiconvex continuous functions.

Proposition 3.1. a) *In the definition of* G*-quasiconvexity the cube* Q *can be replaced by any open bounded set* Ω *such that* $\mid \partial\Omega \mid = 0$.

b) *The function* w *is* G*-quasiconvex if and only if for any* $\mathrm{F} \in \mathrm{G}$ *and* $u \in [\mathrm{G}]_c(Q)$ *we have:*

$$\int_Q w(\mathrm{F})\ dx \ \leq \ \int_Q w(\mathrm{D}u(x)\mathrm{F})\ dx \tag{5}$$

c) *For any* $U \in \mathrm{GL}(n, \mathrm{R})$ *such that* $U\mathrm{G}U^{-1} \subset \mathrm{G}$ *and for any* $\mathrm{W} : \mathrm{G} \to \mathrm{R}$ G*-quasiconvex, the mapping* $\mathrm{W}_U : \mathrm{G} \to \mathrm{R}$, $\mathrm{W}_U(\mathrm{F}) = \mathrm{W}(U\mathrm{F}U^{-1})$ *is* G*-quasi-convex.*

Remark 3.1. The point b) shows that the non-commutativity of the multiplication operation does not affect the definition of G-quasiconvexity. The point c) is a simple consequence of the fact that G is a group.

Proof. The point a) has a straightforward proof by translation and rescaling arguments.

For b) let us consider $\mathrm{F} \in \mathrm{G}$ and an arbitrary open bounded $\Omega \subset \mathrm{R}^n$ with smooth boundary. The application which maps $\phi \in [\mathrm{G}]_c(\Omega)$ to $\mathrm{F}^{-1}\phi\mathrm{F} \in [\mathrm{G}]_c(\mathrm{F}^{-1}(\Omega))$ is well defined and bijective. By a), if the function w is G-quasiconvex then we have

$$\int_{\mathrm{F}^{-1}(\Omega)} w(\mathrm{F}\mathrm{D}(\mathrm{F}^{-1}\phi\mathrm{F})(x))\ \mathrm{d}x \geq \mid \mathrm{F}^{-1}(\Omega) \mid w(\mathrm{F})$$

The change of variables $x = \mathrm{F}^{-1}y$ resumes the proof of b).

With U like in the hypothesis of c), the application which maps $\phi \in [\mathrm{G}]_c(\Omega)$ to $U\phi U^{-1} \in [\mathrm{G}]_c(U^{-1}(\Omega))$ is well defined and bijective. The proof resumes as for the point b). □

The following proposition shows that quasiconvexity in the sense of definition 1.2 is a particular case of G-quasiconvexity.

Proposition 3.2. *Let us consider* $\mathrm{F} \in \mathrm{GL}(n, \mathrm{R})^+$. *Then* w *is* $\mathrm{GL}(n, \mathrm{R})^+$*-quasiconvex in* F *if and only if it is quasiconvex in* F *in the sense of Ball.*

Proof. Let $\mathrm{E} \subset \mathrm{R}^n$ be an open bounded set and $\phi \in [\mathrm{GL}(n, \mathrm{R})^+]_c(\mathrm{E})$. The vector field $\eta = \mathrm{F}(\phi - id)$ verifies the condition that almost everywhere $\mathrm{F} + \mathrm{D}\eta(x)$ is invertible. Therefore, if w is quasiconvex in F, we derive from the inequality:

$$\int_{\mathrm{E}} w(\mathrm{F}\mathrm{D}\phi(y))\ \mathrm{d}y \ \geq \mid \mathrm{E} \mid \mathrm{W}(\mathrm{F}) \quad .$$

We implicitly used the chain of equalities

$$\mathrm{F} + \mathrm{D}\eta(y) \ = \ \mathrm{F} + \mathrm{F}\mathrm{D}\phi(y) \ - \ \mathrm{F} \ = \ \mathrm{F}\mathrm{D}\phi(y) \quad .$$

We have proved that quasiconvexity implies $\mathrm{GL}(n, \mathrm{R})^+$-quasiconvexity.

In order to prove the inverse implication let us consider η such that almost everywhere $F+D\eta(x)$ is invertible. We have therefore $\phi = F^{-1}\psi \in [GL(n,R)^{+}](E)$ and $FD\phi = F + D\eta$. We use now the hypothesis that w is $GL(n,R)^{+}$-quasiconvex in F and we find that w is also quasi-convex. □

Finally, notice that one has to be careful about the domain of definition of a function which has a polyconvex expression.

Proposition 3.3. *The function* $w : GL(n,R) \rightarrow R$ *defined by*

$$w(F) = -\log \mid \det F \mid$$

is not $GL(n,R)$ *quasiconvex.*

Proof. The map has a polyconvex expression. It is not quasiconvex though. To see this fix $\varepsilon \in (0,1)$, $A \in GL(n,R)$ and $\Omega = B(0,1)$. There is a Lipschitz solution to the problem (see Dacorogna-Marcellini Theorem 7.28, Chapter 7.4. [6])

$$\left\{ \begin{array}{ll} Dv(x) \in O(n) & \text{a.e. in } \Omega \\ v(x) = \varepsilon x & x \in \partial\Omega \end{array} \right.$$

We have then, for $u(x) = v(x)/\varepsilon \in [GL(n,R)](\Omega)$:

$$\int_{\Omega} w(ADu(x)) = \int_{\Omega} -\log \mid \det A \mid + \int_{\Omega} n \log \varepsilon < \int_{\Omega} w(A)$$

□

Next proposition justifies this result.

Proposition 3.4. *For any* $w : G \rightarrow R$ *define* $\iota w : G \rightarrow R$ *by:*

$$\iota w(F) = \mid \det F \mid \; w(F^{-1})$$

If w *is* G *quasi-convex then for any* $u \in [G](\Omega)$ *we have:*

$$\int_{\Omega} w(FDu(x)) \geq \int_{\Omega} w(F)$$

Proof. Take u like in the hypothesis. Then for any (continuous) w we have

$$\int_{\Omega} w(Du^{-1}(x)) = \int_{\Omega} \iota w(Du(x))$$

by straightforward computation. Use now the definition 1.4 and the Proposition 2. □

Let us apply this proposition to $w(\mathrm{F}) = -\log \mid \det \mathrm{F} \mid$. Remark that when $\det \mathrm{F}$ goes to zero the function goes to $+\infty$. Now, $\iota w(\mathrm{F}) = \mid \det \mathrm{F} \mid \log \mid \det \mathrm{F} \mid$ and this function can be continuously prolongated to matrices with determinant zero by setting $\iota w(\mathrm{F}) = 0$ if $\det \mathrm{F} = 0$. It is easy to see that the prolongation of ιw is not rank one convex, hence it is not quasiconvex.

4. Proof of Theorem 2.1

The proof is divided into three steps. In the first step we shall prove the following:

(Step 1.) *Let* $w : \mathrm{GL}(n, \mathrm{R}) \to \mathrm{R}$ *be locally Lipschitz. Suppose that for any Lipschitz bounded sequence* $u_h \in [\mathrm{GL}(n, \mathrm{R})]_c^\infty$ *uniformly convergent to* id *on* $\overline{\Omega}$ *and for any* $\mathrm{F} \in \mathrm{GL}(n, \mathrm{R})$ *we have:*

$$\int_\Omega w(\mathrm{F}) \ dx \ \leq \ \liminf_{h \to \infty} \int_\Omega w(\mathrm{F}\mathrm{D}u_h(x)) \ dx \tag{6}$$

Then for any bi-Lipschitz $u : \mathrm{R}^n \to \mathrm{R}^n$ *and for any sequence* $u_h \in [\mathrm{GL}(n, \mathrm{R})]_c^\infty$ *uniformly convergent to* id *on* $\overline{\Omega}$ *we have:*

$$\int_\Omega w(\mathrm{D}u(x)) \ dx \ \leq \ \liminf_{h \to \infty} \int_\Omega w(\mathrm{D}(u_h \circ u)(x)) \ dx \tag{7}$$

Remark 4.1. This is just the point a) of the main theorem for the whole group of linear invertible transformations.

Proof. For $\varepsilon > 0$ sufficiently small consider the set:

$$U^\varepsilon \ = \ \left\{ \mathrm{B} = \overline{\mathrm{B}}(x, r) \subset \Omega \ : \ \exists \ \mathrm{A} \in \mathrm{GL}(n, \mathrm{R}) \ , \ \int_\mathrm{B} \mid \mathrm{D}u(x) - \mathrm{A} \mid < \varepsilon \mid \mathrm{B} \mid \right\}$$

From the Vitali covering theorem and from the fact that u is bi-Lipschitz we deduce that there is a sequence $\mathrm{B}_j = \overline{\mathrm{B}}(x_j, r_j) \in U^\varepsilon$ such that:

- $\mid \Omega \setminus \bigcup_j \mathrm{B}_j \mid = 0$
- for any j u is approximatively differentiable in x_j and $\mathrm{D}u(x_j) \in \mathrm{GL}(n, \mathrm{R})$
- we have

$$\int_{\mathrm{B}_j} \mid \mathrm{D}u(x) - \mathrm{D}u(x_j) \mid < \ \varepsilon \mid \mathrm{B}_j \mid$$

Choose N such that

$$\mid \Omega \setminus \bigcup_{j=1}^N \mathrm{B}_j \mid < \ \varepsilon$$

We have therefore:

$$\int_\Omega w(\mathrm{D}(u_h \circ u)(x)) \ \geq \ \sum_{j=1}^N \int_{\mathrm{B}_j} w(\mathrm{D}(u_h \circ u)(x)) \ - \ \mathrm{C}\varepsilon$$

$$\sum_{j=1}^{N} \int_{\mathrm{B}_j} w(\mathrm{D}(u_h \circ u)(x)) = \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3$$

where the quantities J_i are given below, with their estimates.

$$\mathrm{J}_1 = \sum_{j=1}^{N} \int_{\mathrm{B}_j} [w(\mathrm{D}u_h(u(x))\mathrm{D}u(x)) - w(\mathrm{D}u_h(u(x))\mathrm{D}u(x_j))]$$

$$\mid \mathrm{J}_1 \mid \leq \sum_{j=1}^{N} \int_{\mathrm{B}_j} \mid w(\mathrm{D}u_h(u(x))\mathrm{D}u(x)) - w(\mathrm{D}u_h(u(x))\mathrm{D}u(x_j)) \mid < \mathrm{C}\varepsilon$$

$$\mathrm{J}_2 = \sum_{j=1}^{N} \int_{\mathrm{B}_j} [w(\mathrm{D}u_h(u(x))\mathrm{D}u(x_j)) - w(\mathrm{D}u_h(\overline{u}_j(x))\mathrm{D}u(x_j))]$$

where $\bar{u}_j(x) = u(x_j) + \mathrm{D}u(x_j)(x - x_j)$. We have the estimate:

$$\mid \mathrm{J}_2 \mid \leq \mathrm{C}\varepsilon$$

Indeed, by changes of variables we can write:

$$\mathrm{I}'_j \ = \ \int_{\mathrm{B}_j} w(\mathrm{D}u_h(u(x))\mathrm{D}u(x_j)) \ = \ \int_{u(\mathrm{B}_j)} w(\mathrm{D}u_h(y)\mathrm{D}u(x_j) \ \mid \det \mathrm{D}u^{-1}(y) \mid$$

$$\mathrm{I}''_j \ = \ \int_{\mathrm{B}_j} w(\mathrm{D}u_h(\bar{u}_j(x))\mathrm{D}u(x_j)) \ = \ \int_{\bar{u}_j(\mathrm{B}_j)} w(\mathrm{D}u_h(y)\mathrm{D}u(x_j) \ \mid \det(\mathrm{D}u(x_j))^{-1} \mid$$

The difference $\mid \mathrm{I}'_j - \mathrm{I}''_j \mid$ is majorised like this

$$\mid \mathrm{I}'_j - \mathrm{I}''_j \mid \leq \int_{u(\mathrm{B}_j)\cap\bar{u}_j(\mathrm{B}_j)} \mathrm{C} \mid\mid \det \mathrm{D}u^{-1}(y) \mid - \mid \det(\mathrm{D}u(x_j))^{-1} \mid\mid \ + \mathrm{C} \mid u(\mathrm{B}_j)\Delta\bar{u}_j(\mathrm{B}_j) \mid$$

The function $\mid \det \cdot \mid$ is rank one convex and satisfies the growth condition $\mid \det \mathrm{F} \mid \leq c(1+ \mid \mathrm{F} \mid^n)$ for any $\mathrm{F} \in \mathrm{GL}(n, \mathrm{R})$. Therefore this function satisfies also the inequality:

$$\mid\mid \det \mathrm{F} \mid - \mid \det \mathrm{P} \mid\mid \leq \ \mathrm{C} \mid \mathrm{F} - \mathrm{P} \mid \left(1+ \mid \mathrm{F} \mid^{n-1} + \mid \mathrm{P} \mid^{n-1}\right)$$

Use now this inequality, the properties of the chosen Vitali covering and the uniform bound on Lipschitz norm of u , u_h , to get the claimed estimate.

$$\mathrm{J}_3 = \sum_{j=1}^{N} \int_{\mathrm{B}_j} w(\mathrm{D}u_h(\overline{u}_j(x))\mathrm{D}u(x_j))$$

By the change of variable $y = \overline{u}_j(x)$ and the hypothesis we have

$$\liminf_{h\to\infty} \mathrm{J}_3 \geq \ \liminf_{h\to\infty} \sum_{j=1}^{N} \int_{\mathrm{B}_j} w(\mathrm{D}u(x_j))$$

Put all the estimates together and pass to the limit with $N \to \infty$ and then $\varepsilon \to 0$. □

(Step 2.) If we replace in **Step 1.** *the group* $\mathrm{GL}(n, \mathrm{R})$ *by a Lie subgroup* G *the conclusion is still true.*

Proof. Indeed, remark that in the proof of the previous step it is used only the fact that $\mathrm{GL}(n, \mathrm{R})$ is a group of invertible maps. □

Step 3. *The point b) of the Theorem 2.1 is true.*

Remark 4.2. In the classical setting of quasiconvexity, this step is proven by an argument involving Lipschitz extensions with controlled Lipschitz norm. This is not known to be true in the realm of bi-Lipschitz maps. That is why we shall use a different approach.

Proof. Because G is a group, it is sufficient to make the proof for $\mathrm{F} = 1$.

Let $u_h \in [\mathrm{G}]_c$ be a sequence weakly * convergent to id on Ω and $\mathrm{D} \subset\subset \Omega$. For $\varepsilon > 0$ sufficiently small and $\mathrm{C} > 1$ we have

$$\mathrm{D}_{\mathrm{C}\varepsilon} \ = \ \bigcup_{x \in \mathrm{D}} \mathrm{B}(x, \mathrm{C}\varepsilon) \ \subset \ \Omega$$

It is not restrictive to suppose that

$$\lim_{h \to \infty} \int_{\Omega} w(\mathrm{D}u_h) \, \mathrm{d}x$$

exists and it is finite. For any $\varepsilon > 0$ there is N_ε such that for any $h > N_\varepsilon$ $u_h(\mathrm{D}) \subset \mathrm{D}_\varepsilon$.

Take a minimal Lipschitz extension

$$\overline{u}_h : \mathrm{D}_{\mathrm{C}\varepsilon} \setminus \mathrm{C} \to \mathrm{R}^n \ , \ \overline{u}_h(x) = \begin{cases} u_h(x) & , \ x \in \partial \mathrm{D} \\ x & , \ x \in \partial \mathrm{D}_{\mathrm{C}\varepsilon} \end{cases}$$

The Lipschitz norm of this extension, denoted by k_h, is smaller than some constant independent on h.

Now, for any h define:

$$\psi_h \ = \ \frac{1}{2k_h} \, \overline{u}_{h_{|\mathrm{D}_{\mathrm{C}\varepsilon} \setminus \mathrm{D}}}$$

According to Dacorogna-Marcellini Theorem 7.28, Chapter 7.4. [6], there is a solution σ_h of the problem

$$\begin{cases} \mathrm{D}\sigma_h \in \mathrm{O}(n) & \text{a. e. in } \mathrm{D}_{\mathrm{C}\varepsilon} \setminus \mathrm{D} \\ \sigma_h \ = \ \psi_h & \text{on } \partial(\mathrm{D}_\varepsilon \setminus \mathrm{D} \end{cases}$$

Let

$$v_h(x) \;=\; \begin{cases} u_h(x) & x \in \mathrm{D} \\ k_h\sigma_h(x) & x \in \Omega \setminus \mathrm{D} \end{cases}$$

Note that $\mathrm{D}v_h \in \mathrm{CO}(n)$.

The following estimate is then true:

$$\mid \int_{\mathrm{D}} w(\mathrm{D}u_h)\ \mathrm{d}x \;-\; \int_{\Omega} w(\mathrm{D}v_h)\ \mathrm{d}x \mid = \mid \int_{\mathrm{D}_{\mathrm{C}\varepsilon} \setminus \mathrm{D}} w(\mathrm{D}v_h)\ \mathrm{d}x \mid \leq$$

$$\leq \int_{\mathrm{D}_{\mathrm{C}\varepsilon} \setminus \mathrm{D}} \mid w(\mathrm{D}v_h) \mid \ \mathrm{d}x \;\leq\; \mathrm{C} \mid \mathrm{D}_\varepsilon \setminus \mathrm{D} \mid$$

w is G-quasiconvex, therefore:

$$\int_{\mathrm{D}_\varepsilon} w(\mathrm{D}v_h)\ \mathrm{d}x \;\geq \mid \mathrm{D}_\varepsilon \mid w(1)$$

We put all together and we get the inequality:

$$\lim_{h\to\infty} \int_{\mathrm{D}} w(\mathrm{D}u_h)\ \mathrm{d}x \;\geq \mid \mathrm{D}_\varepsilon \mid w(1) - \mathrm{C} \mid \mathrm{D}_\varepsilon \setminus \mathrm{D} \mid$$

The proof finishes after we pass ε to 0. □

5. Application: a class of quasiconvex functions

The goal of this section is to give a class of quasi-convex isotropic functions which seem to be complementary to the polyconvex isotropic ones. We quote the following result of Thompson and Freede [15], Ball [2], Le Dret [11].

Theorem 5.1. *Let $g : [0,\infty)^n \to \mathrm{R}$ be convex, symmetric and nondecreasing in each variable. Define the function w by*

$$w : gl(n, \mathrm{R}) \to \mathrm{R}\ , \quad w(\mathrm{F}) = g(\sigma(\mathrm{F})).$$

Then w is convex.

We shall use the Theorem 6.2. Buliga [5]. We need a notation first. Let $x = (x_1, ..., x_n) \in \mathrm{R}^n$ be a vector. Then the vector $x^\downarrow = (x_1^\downarrow, ..., x_n^\downarrow) \in \mathrm{R}^n$ is obtained by rearranging in decreasing order the components of x . Remark that for any symmetric function $h : \mathrm{R}^n \to \mathrm{R}$ there exists and it is unique the function $p : \mathrm{R}^n \to \mathrm{R}$ defined by the relation:

$$p(\sum_{i=1}^{k} x_i^\downarrow) \;=\; h(x_k)$$

Theorem 5.2. *Let* $g : (0, \infty)^n \to \mathrm{R}$ *be a continuous symmetric function and* $h : \mathrm{R}^n \to \mathrm{R}$, $h = g \circ \exp$. *Consider also the function* $p : \mathrm{R}^n \to \mathrm{R}$, *previously defined, associated to the symmetric function* h .

Suppose that:

(a) h *is convex,*

(b) p *is nonincreasing in each argument.*

Let $\Omega \subset \mathrm{R}^n$ *be bounded, with piecewise smooth boundary and* $\phi : \overline{\Omega} \to \mathrm{R}$ *be any Lipschitz function such that* $\mathrm{D}\phi(x) \in \mathrm{GL}(n, \mathrm{R})^+$ *a.e. and* $\phi(x) = x$ *on* $\partial\Omega$. *Define the function*

$$w : \mathrm{GL}(n, \mathrm{R})^+ \to \mathrm{R} \, , \quad w(\mathrm{F}) = g(\sigma(\mathrm{F}))$$

Then for any $\mathrm{F} \in \mathrm{GL}(n, \mathrm{R})^+$ *we have:*

$$\int_\Omega w(\mathrm{F}\mathrm{D}\phi(x)) \ \geq | \, \Omega \, | \, w(\mathrm{F}) \tag{8}$$

A consequence of Theorem 5.2 and Theorem 2.1 (a) is:

Proposition 5.1. *In the hypothesis of Theorem 5.2, let* $\phi_h : \Omega \to \mathrm{R}^n$ *be a sequence of Lipschitz bounded functions such that*

(a)*for any* h $\mathrm{D}\phi_h(x) \in \mathrm{GL}(n, \mathrm{R})^+$ *a.e. in* Ω .

(b)*the sequence* ϕ_h *converges uniformly to* $u : \Omega \to \Omega$, *bi-Lipschitz function.*

Then

$$\liminf_{h \to \infty} \int_\Omega w(\mathrm{D}\phi_h(x)) \ \geq \ \int_\Omega w(\mathrm{D}u(x)) \tag{9}$$

Proof. It is clear that Theorem 5.2 implies the hypothesis of point (a), Theorem 2.1. Indeed, the conclusion of Theorem 5.2 can be written like this: for any $u \in [\mathrm{GL}(n, \mathrm{R})^+](\Omega)$ such that

$$\bar{\mathrm{D}}u(\Omega) = \frac{1}{| \, \Omega \, |} \int_\Omega \mathrm{D}u(x) \ \mathrm{d}x \ \in \mathrm{GL}(n, \mathrm{R})^+$$

we have the inequality

$$\int_\Omega w(\mathrm{D}u(x)) \ \mathrm{d}x \geq \int_\Omega w(\bar{\mathrm{D}}u(\Omega)) \ \mathrm{d}x$$

Take a sequence of mapping $(u_h) \subset [\mathrm{GL}(n, \mathrm{R})^+](\Omega)$ uniformly convergent to $\mathrm{F} \in \mathrm{GL}(n, \mathrm{R})^+$. The previous inequality and the continuity of w imply:

$$\int_\Omega w(\mathrm{F}) \ \mathrm{d}x \ \leq \ \int_\Omega w(\mathrm{D}u_h(x)) \ \mathrm{d}x$$

Apply now Theorem 2.1 (a) and obtain the thesis. □

We close with two examples of functions which we can prove that are $\mathrm{GL}(n,\mathrm{R})^+$ quasiconvex using Theorem 5.2.

For the first example we begin by making the notation $\mathrm{F} = \mathrm{R_F} U_\mathrm{F}$ for the polar decomposition of $\mathrm{F} \in \mathrm{GL}(n,\mathrm{R})^+$, with U_F symmetric and positive definite. The first example is then the function:

$$w : \mathrm{GL}(n,\mathrm{R})^+ \rightarrow \mathrm{R} \ , \quad w(\mathrm{F}) = \det \mathrm{F} \ \log\left(\mathit{trace}\ U_\mathrm{F}\right)$$

With the notation introduced in Proposition 3.4, let's look to the the function $\hat{w} = \iota w$. It has the expression:

$$\hat{w} : \mathrm{GL}(n,\mathrm{R})^+ \rightarrow \mathrm{R} \ , \quad \hat{w}(\mathrm{F}) \ = \ \log\left(\mathit{trace}\ U_\mathrm{F}^{-1}\right)$$

It is a matter of straightforward computation to check that $\hat{w}$ verifies the hypothesis of Theorem 5.2. It is therefore $\mathrm{GL}(n,\mathrm{R})^+$ quasiconvex. By Proposition 3.4 w is $\mathrm{GL}(n,\mathrm{R})^+$ quasiconvex, too, hence lower semicontinuous in the sense of Theorem 2.1 (a).

For the second example set

$$\| \mathrm{F} \|_k \ = \ \left(\prod_{i=1}^{k} \sigma_i^{\downarrow}(\mathrm{F}) \right)^{1/k}$$

and define:

$$w(\mathrm{F}) \ = \ \sum_{i=1}^{n} \frac{1}{\| \mathrm{F} \|_i^{\alpha}}$$

for some $\alpha \geq 2$. The associated function h is then

$$h(x_1, ..., x_n) \ = \ \sum_{k=1}^{n} \exp\left((-\alpha/k) \sum_{i=1}^{k} x_i^{\downarrow} \right)$$

which again satisfies the hypothesis of the Theorem 5.2.

The class of functions w described in Theorem 5.2 and the class of polyconvex functions seem to be different. Further investigations are needed in order to make stronger assertions. Notice that by picking h linear we obtain a polyconvex function, like

$$w(\mathrm{F}) = -\log \mid \det \mathrm{F} \mid$$

We have seen in Proposition 3.3 that this function is not $\mathrm{GL}(n,\mathrm{R})$ quasiconvex but Proposition 5.1 tells that w is $\mathrm{GL}(n,\mathrm{R})^+$ quasiconvex.

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Marius Buliga
Institute of Mathematics
of the Romanian Academy
PO BOX 1-764
RO 70700, Bucharest
Romania
e-mail: Marius.Buliga@imar.ro
and
Institut Bernoulli
Bâtiment MA
Ecole Polytechnique Fédérale
1015 Lausanne
Switzerland
e-mail: Marius.Buliga@epfl.ch