The Magnetic Moyal algebras

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In colaboration with Viorel Iftimie, Marius Mantoiu and Serge Richard

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Using these techniques we proved a number of spectral results for quantum Hamiltonians in magnetic fields.

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- These two descriptions may be put together in a strict deformation quantization in the sense of M. Rieffel.
- In colaboration with Marius Măntoiu and Serge Richard we have defined and studied associated coherent states quantization and Bargmann representation.

Contents

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  with momentum space $\mathcal{X}'$, the dual of $\mathcal{X}$,
  (canonically isomorphic to $\mathbb{R}^d$).
- The canonical symplectic form on Ξ:
  $\sigma\big((x,\xi),(y,\eta)\big) :=< \xi, y > - < \eta, x >$
  with $< ., . >$ the duality application $\mathcal{X}' \times \mathcal{X} \to \mathbb{R}$.

The magnetic field

- The magnetic field is described by a closed 2-form B on $\mathcal{X}$:

$$B = \sum_{j,k=1}^{n} B_{jk}(x) dx_j \wedge dx_k, \quad B_{jk}(x) = -B_{kj}(x), \quad dB = 0.$$

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$$A - A' = \nabla\Phi \quad \Leftrightarrow \quad dA = dA' = B.$$
- If B has components of class $C^\infty_{\text{pol}}(\mathcal{X})$, then the following formula always provides a vector potential with components of class $C^\infty_{\text{pol}}(\mathcal{X})$:
$$A_j(x) := -\sum_{k=1}^{n} \int_0^1 ds\, B_{jk}(sx) s x_k.$$

The magnetic field - the classical picture

- In the Hamiltonian formalism, the Lorentz force can be described by replacing the usual canonical pair of variables

  $$(x, \xi) \text{ on } \Xi \text{ by the pair of variables } (x, \xi + A(x))$$

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defined once we have chosen a vector potential A for B.
- Appearently this prescription is highly non-unique due to the gauge ambiguity.
- But, one can easily see that the Hamilton equations of motion only depend on the magnetic field B, through the usual Lorentz force term:

$$\left(\partial_t^2 x_j\right)(t) = \sum_{1 \leq k \leq d} B_{j,k}(x(t))\left(\partial_t x_k\right)(t).$$

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where $\mathfrak{j}_B$ is the canonical isomorphism

$$\mathfrak{j}_B : \Xi \to \Xi^*, \quad < \mathfrak{j}_B(X), Y >:= \sigma^B(X, Y).$$

The gauge invariant formalism

Using the canonical global coordinates we have:

$$\{f,g\}^B(x,\xi):=$$

$$=\sum_{j=1}^n\left[\left(\partial_{\xi_j}f\right)(x,\xi)\left(\partial_{x_j}g\right)(x,\xi)-\left(\partial_{x_j}f\right)(x,\xi)\left(\partial_{\xi_j}g\right)(x,\xi)\right]+$$

$$\sum_{j,k=1}^n B_{jk}(x)\left(\partial_{\xi_j}f\right)(x,\xi)\left(\partial_{\xi_k}g\right)(x,\xi)$$

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- Consists in a complex Hilbert space $\mathcal{H}$
- and two strongly continuous unitary representations:

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- satisfying the Weyl commutation relations:

$$U(x)V(\xi) = e^{i\xi(x)}\ V(\xi)U(x), \qquad x \in \mathcal{X},\ \xi \in \mathcal{X}'.$$

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The Weyl system - symplectic form

- Is given by a complex Hilbert space $\mathcal{H}$
- and a strongly continuous map

$$\Xi \ni X \mapsto W(X) \in \mathcal{U}(\mathcal{H}),$$

- satisfying the relations

$$W(X)W(Y) = \exp\left\{\frac{i}{2}\sigma(X,Y)\right\} W(X+Y), \quad W(0) = 1.$$

(just take $W(x,\xi) := e^{(i/2)\xi(x)} U(-x) V(\xi)$)

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The quantum observables

- For any test function $\phi \in \mathcal{S}(\Xi;\mathbb{R})$
- we can define the associated *quantum observable*

$$\mathfrak{Op}(\phi) := \int_{\Xi} [\mathcal{F}^{-1}\phi](X)\, W(X)\, dX \ \in \mathbb{B}(\mathcal{H})$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform on $\mathcal{S}(\Xi)$.

And we can extend this formula by duality to $\mathcal{S}'(\Xi;\mathbb{R})$.

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In order to obtain the quantum description of systems in magnetic fields, the 'paradigm' is to quantize the system with the usual canonical variables (x, ξ) replaced by the 'magnetic' canonical variables' $(x, \xi - A(x))$ with A a vector potential for the magnetic field B. Thus

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- Suppose chosen a gauge A for the magnetic field B.
- On $L^2(\mathcal{X})$ we define $$Q_1, \ldots, Q_n,$$ the multiplication with the variables, and $$\Pi_1^A := D_1 - A_1, \ldots, \Pi_n^A := D_n - A_n, \quad \text{with } D_j := -i\partial_j$$ representing the canonical momenta in the magnetic field.

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  representing the canonical momenta in the magnetic field.
- While for the usual Hamiltonian $h(x, \xi) = (\xi^2/2) + V(x)$ the quantization is rather clear (and by chance gauge covariant) things are rather difficult for some 'general' Hamiltonians (relativistic, effective Hamiltonians) and other observables of the system.

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Unfortunately this procedure produces operators that are no longer gauge covariant! (except the case discussed before).

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They satisfy: $U^A(x)U^A(y) = \Omega^B(Q;x,y)U^A(x+y)$.
with $\Omega^B(Q;x,y) := \exp\left(-i\int_{<Q,Q+x,Q+x+y>} B\right)$.

Defining now

$$W^A((x,\xi)) := e^{-i<\xi,x/2>} V^A(\xi) U^A(x) =$$

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Thus

$$W^A(X)W^A(Y) = e^{i\sigma(X,Y)} \frac{\Omega^B(Q;x,y)}{\Omega^B(Q;y,x)} W^A(Y)W^A(X).$$

The magnetic Weyl calculus - definition

- For any test function $f : \Xi \to \mathbb{C}$ we define the associated magnetic Weyl operator:
$$\mathfrak{Op}^A(f) := \int_\Xi dX \hat{f}(X) W^A(X) \in \mathbb{B}[\mathcal{H}]$$
that leaves $\mathcal{S}(\mathcal{X})$ invariant [M.P., *J. Math. Phys. 04*].

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Observation: *Gauge covariance*

The Schrödinger representations associated to any two gauge-equivalent vector potentials are unitarily equivqlent:
$$A' = A + d\varphi \quad \Rightarrow \quad \mathfrak{Op}^{A'}(f) = e^{i\varphi(Q)} \mathfrak{Op}^A(f) e^{-i\varphi(Q)}.$$

Integral kernels associated to Weyl symbols

$\mathfrak{Op}^A(f)$ is an integral operator having the following integral kernel that can be defined in terms of f

$$\mathfrak{Op}^A(f) := \mathfrak{Int}(K^A f), \quad \big[\big(\mathfrak{Int}(\Phi)\big)u\big](x) := \int \Phi(x,y)u(y)dy,$$

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with

$$K^A f := \Lambda^A \Theta^{-1} \mathfrak{F}^- f;$$

- $\Lambda^A(x,y) := \exp\left(-i\int_{[x,y]} A\right)$,
- $\big(\mathfrak{F}^- f\big)(x,y) := (2\pi)^{-n}\int_{\mathcal{X}'} d\eta e^{i\eta\cdot y} f(x,\eta)$,
- $\big(\Theta f\big)(x,y) := f(x-y/2, x+y/2), \quad \big(\Theta^{-1} g\big)(x,y) = g\left(\frac{x+y}{2}, y-x\right)$.

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We denote by $$K^0 f := \Theta^{-1}\mathfrak{F}^- f.$$

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and notice that it has the integral kernel

$$\begin{aligned}\big[\widetilde{K^A}(f)\big](x,y) &= \Lambda^A(0,x)\Lambda^A(y,0)\Lambda^A(x,y)\big[K^0(f)\big](x,y)\\ &= \omega^B(0,x,y)\big[K^0(f)\big](x,y).\end{aligned}$$

with $\omega^B(0,x,y) := \exp\left(-i\int_{<0,x,y>} B\right)$.

The 'standard' representations - unitary equivalence

- Let us choose some $q \in \mathcal{X}$ and denote by $\Lambda^A_q(x) := \Lambda^A(q,x)$ and

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- Then, denoting by $\omega^B_{0,q}(x) := \omega^B(0,q,x)$, we have

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- **We have obtained a class of unitary equivalent representations indexed by the points in $\mathcal{X}$, depending only on the magnetic field B. We shall use the notation $\widetilde{\mathfrak{Op}}^B(f) := \widetilde{\mathfrak{Op}^A}(f)$.**

The magnetic Moyal algebra

The *magnetic* Moyal product

Definition

The above 'magnetic' functional calculus induces a *magnetic composition* on the complex linear space of test functions $\mathcal{S}(\Xi)$:

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Explicitly we have:

$$(f\sharp^B g)(X) := 4^n \int_\Xi dY \int_\Xi dZ\, e^{-i\int_{\mathcal{T}_X(Y,Z)} \sigma^B} f(X-Y)\, g(X-Z)$$

where $\mathcal{T}_X(Y,Z)$ is the triangle in Ξ having vertices:

$$X-Y-Z, \quad X+Y-Z, \quad X-Y+Z.$$

The *magnetic* Moyal product

Theorem [M.P., *J. Math. Phys. 04*]

For a magnetic field B with components of class $C^\infty_{\text{pol}}(\mathcal{X})$, the composition $\sharp^B$ defines a bilinear map

$$\mathcal{S}(\Xi) \times \mathcal{S}(\Xi) \ni (\phi, \psi) \mapsto \phi \sharp^B \psi \in \mathcal{S}(\Xi)$$

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Proposition [M.P., *J. Math. Phys. 04*]

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$$\int_\Xi (\phi \sharp^B \psi)(X)\, dX = \int_\Xi \phi(X)\, \psi(X)\, dX, \qquad \forall (\phi, \psi) \in \Big(\mathcal{S}(\Xi)\Big)^2,$$

$$\int_\Xi (\phi \sharp^B \psi)(X) \chi(X) dX = \int_\Xi \phi(X) (\psi \sharp^B \chi)(X) dX, \quad \forall (\phi, \psi, \chi) \in \Big(\mathcal{S}(\Xi)\Big)^3.$$

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We ca extend the product $\sharp^B$ by duality to bilinear maps:

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We set:

$$\mathfrak{M}^B(\Xi) := \left\{ F \in \mathcal{S}'(\Xi) \mid F\sharp^B\phi \in \mathcal{S}(\Xi), \phi\sharp^B F \in \mathcal{S}(\Xi), \forall\phi \in \mathcal{S}(\Xi) \right\}$$

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and the usual complex conjugation as *-*conjugation*.
Moreover, the above algebraic structures may be organized as a **strict deformation quantization of the algebra of observables in the sense of Rieffel.** [M.P. 05]

The topology

Theorem [M.P. 11]

- For a magnetic field B with components of class $C^\infty_{\text{pol}}(\mathcal{X})$, we have two natural linear applications

$$\mathfrak{M}^B(\Xi) \to \mathbb{B}\Big(\mathcal{S}(\Xi)\Big), \qquad \mathfrak{M}^B(\Xi) \to \mathbb{B}\Big(\mathcal{S}'(\Xi)\Big).$$

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Proposition [M.P. 11]

The topologies induced on $\mathfrak{M}^B(\Xi)$ by restriction from $\mathcal{S}'(\Xi)$ are coarser then the above locally convex topology.

The norm

- The familly:

$$\mathfrak{C}^B(\Xi) := \left\{ F \in \mathcal{S}'(\Xi) \mid \mathfrak{Op}^A(F) \in \mathbb{B}[L^2(\mathcal{X})] \right\}$$

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- $\mathfrak{C}^B(\Xi)$ is a C^*-algebra isomorphic to $\mathbb{B}[L^2(\mathcal{X})]$.

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Hörmander type symbols

For $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$ we define $\forall F \in C^\infty(\Xi)$ the seminorms

$$|F|^{(m;\rho,\delta)}_{(a,\alpha)} := \sup_{(x,\xi)\in\Xi} <\xi>^{-m+\rho|\alpha|-\delta|a|} \left|(\partial^a_x \partial^\alpha_\xi F)(x,\xi)\right|,$$

and the Fréchet space

$$S^m_{\rho,\delta}(\Xi) := \left\{F \in C^\infty(\Xi) \mid \forall (a,\alpha),\ |F|^{(m;\rho,\delta)}_{(a,\alpha)} < \infty\right\}.$$

Proposition [I.M.P., *Proc. RIMS 07*]

For $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$ we have $S^m_{\rho,\delta}(\Xi) \subset \mathfrak{M}^B(\Xi)$.

Symbols of 1-dimensional projections

Given a 1-dimensional orthogonal projection

$$P_\phi := |\phi >< \phi| \text{ for some } \phi \in L^2(\mathcal{X}) \text{ with } \|\phi\|_2 = 1$$

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- Notice that the magnetic operator associated to the above symbol is no longer a 1-dimensional projection!
- Notice further that taking $\phi^A := \left(\Lambda^A_\bullet\right)^{-1}\phi$ we have that

$$\left(\Lambda^A\right)^{-1}\left(\phi^A \otimes \overline{\phi^A}\right) = \overline{\omega^B}(\phi \otimes \overline{\phi}).$$

The magnetic symbols of 1-dimensional projections

Definition

Let us put $p^B_\phi := \mathfrak{F}\Theta\left[\overline{\omega^B}(\phi \otimes \overline{\phi})\right] = \mathfrak{F}\Theta\left(\Lambda^A\right)^{-1}\left[(\phi^A \otimes \overline{\phi^A})\right]$.

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Let us put $p_\phi^B := \mathfrak{F}\Theta\left[\overline{\omega^B}(\phi\otimes\overline{\phi})\right] = \mathfrak{F}\Theta\left(\Lambda^A\right)^{-1}\left[\left(\phi^A\otimes\overline{\phi^A}\right)\right]$.

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and

$$\mathfrak{Op}^A(p_\phi^B) = \mathfrak{Int}\left[\Lambda^A\Theta^{-1}\mathfrak{F}^-\mathfrak{F}\Theta\left[\overline{\omega^B}(\phi\otimes\overline{\phi})\right]\right]$$
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Thus the state $|\phi><\phi|$ in the magnetic field B has associated an idempotent real symbol p^B_ϕ.

The magnetic Weyl calculus

'Magnetic' pseudo-differential operators

Definition

Choosing any vector potential A for B we define the associated classes of *magnetic* pseudodifferential operators on $\mathcal{H} := L^2(\mathcal{X})$ with Hörmander type symbols:

$$\boldsymbol{\Psi}^m_{\rho,\delta}(A) := \mathfrak{Op}^A[S^m_{\rho,\delta}(\Xi)].$$

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Theorem [I.M.P., *Proc. RIMS 07*]

If the magnetic field B has components of class $C^\infty_{\text{pol}}(\mathcal{X})$, for any m_1 and m_2 in $\mathbb{R}$ and for any $0 \leq \delta \leq \rho \leq 1$ we have:

$$S^{m_1}_{\rho,\delta}(\Xi) \sharp^B S^{m_2}_{\rho,\delta}(\Xi) \subset S^{m_1+m_2}_{\rho,\delta}(\Xi).$$

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Under the above hypothesis on the magnetic field B, for any vector potential A we have that in the Schrödinger representation:

$$\mathbf{\Psi}^{m_1}_{\rho,\delta}(A) \cdot \mathbf{\Psi}^{m_2}_{\rho,\delta}(A) \subset \mathbf{\Psi}^{m_1+m_2}_{\rho,\delta}(A).$$

L^2-continuity

Theorem [I.M.P., *Proc. RIMS 07*]

If the magnetic field B has components of class $BC^\infty(\mathcal{X})$, then $S^0_{\rho,\rho}(\Xi)$, with $0 \leq \rho < 1$ and $S^0_{\rho,\delta}(\Xi)$, with $0 \leq \delta < \rho \leq 1$ are contained in $\mathfrak{C}^B(\Xi)$ and there exist two constants $c(n) \in \mathbb{R}_+$ and $p(n) \in \mathbb{N}$, depending only on the dimension n of the space $\mathcal{X}$, such that we have the estimation:

$$\|F\|_B \leq c(n)|F|_{(p(n),p(n))}.$$

where

$$|F|_{(p,q)} := \max_{|a|\leq p} \max_{|\alpha|\leq q} \sup_{(x,\xi)\in\Xi} \left|\left(\partial_x^a \partial_\xi^\alpha F\right)(x,\xi)\right|$$

are the seminorms defining the topology of $S^0_{\rho,\delta}(\Xi)$.

Sobolev spaces

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- For any $m > 0$ we define:

$$\wp_m(x,\xi) :=< \xi >^m \equiv (1+|\xi|^2)^{m/2}$$

so that $\wp_m \in S^m_{1,0}(\Xi) \subset \mathfrak{M}^B(\Xi)$ and for any potential vector A we can define:

$$\mathfrak{p}^A_m := \mathfrak{O}\mathfrak{p}^A(\wp_m).$$

Sobolev spaces

Definition

Suppose that the magnetic field B has components of class $BC^\infty(\mathcal{X})$ and suppose chosen a vector potential A for it. For any $m > 0$ we define the complex linear space:

$$\mathcal{H}^m_A(\mathcal{X}) := \left\{ u \in L^2(\mathcal{X}) \mid \mathfrak{p}^A_m u \in L^2(\mathcal{X}) \right\}.$$

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Proposition [I.M.P., *Proc. RIMS 07*]

The space $\mathcal{H}^m_A(\mathcal{X})$ is a Hilbert space for the scalar product:

$$< u, v >_{(m,A)} := (\mathfrak{p}^A_m u, \mathfrak{p}^A_m v)_2 + (u, v)_2.$$

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Suppose that the magnetic field B has components of class $BC^\infty(\mathcal{X})$ and suppose chosen a vector potential A. For any $m > 0$ we define the space $\mathcal{H}_A^{-m}(\mathcal{X})$ as the dual space of $\mathcal{H}_A^m(\mathcal{X})$ with the dual norm:

$$\|\phi\|_{(-m,A)} := \sup_{u \in \mathcal{H}_A^m(\mathcal{X}) \setminus \{0\}} \frac{| < \phi, u > |}{\|u\|_{(m,A)}}$$

that induces a scalar product.

We also denote $\mathcal{H}_A^0(\mathcal{X}) := L^2(\mathcal{X})$.

Elliptic symbols

Definition

For $m > 0$ a symbol $F \in S^m_{\rho,\delta}(\Xi)$ is said to be elliptic if there exist two positive constants R and C such that for any $(x, \xi) \in \Xi$ with $|\xi| \geq R$ one has that

$$|F(x, \xi)| \geq C < \xi >^m$$

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- If $F \geq 0$ then $\mathfrak{Op}^A(F)$ is lower semibounded and we have a strong Gårding inequality.
- If A is chosen in $C^\infty_{\text{pol}}(\mathcal{X})$, then $\mathfrak{Op}^A(F)$ is essentially self-adjoint on $\mathcal{S}(\mathcal{X})$.

Extension of the Weyl calculus

- Let us denote by $\mathbb{B}[\mathcal{V}_1;\mathcal{V}_2]$ the space of continuous operators from the locally convex space $\mathcal{V}_1$ to the locally convex space $\mathcal{V}_2$.

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- Let us denote by $F_a^{-1}(\xi) := 1/F_a(\xi)$ its usual inverse (for pointwise multiplication).

An inversion result

- Let $m > 0$ and $F \in S^m_{1,0}(\Xi)$ be an elliptic symbol that does not depend on the $x \in \mathcal{X}$ variable.
- Let $-a < \inf_{\xi \in \mathcal{X}'} F(\xi)$ and $F_a(\xi) := F(\xi) + a$.
- Let us denote by $F_a^{-1}(\xi) := 1/F_a(\xi)$ its usual inverse (for pointwise multiplication).
- We define: $\mathfrak{r}_a^B[F] := F_a \sharp^B F_a^{-1} - 1 \in \mathfrak{M}^B(\Xi)$.

An inversion result

Theorem [M.P.R., *J.Func. Anal. 07*]

Suppose that the magnetic field B has components of class $BC^\infty(\mathcal{X})$ and let $m > 0$, $F \in S^m_{1,0}(\Xi) \cap C^\infty(\mathcal{X}')$ be elliptic and $a \in \mathbb{R}_+$ large enough. Then F_a has an inverse for the $\sharp^B$ product, F_a^- in $\mathfrak{C}^B(\mathcal{X})$ and this inverse is given by the formula

$$F_a^- = F_a^{-1} \sharp^B \left(\sum_{k \in \mathbb{N}} (\mathfrak{r}_a^B[F])^{\sharp^B k} \right)$$

with the series converging in the C^*-norm $\|.\|_B$.

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- Thus let us define $\mathfrak{s}_0 := 1$

$$\mathfrak{s}_m := \wp_{m,a_m}, \quad \text{for } m > 0; \quad \mathfrak{s}_m := \wp^-_{|m|,a_{|m|}}, \quad \text{for } m < 0$$

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- For any $X \in \Xi$ let us define

$$\mathfrak{l}_X(Y) := \sigma(X, Y), \qquad \mathfrak{ad}^B_X[F] := \mathfrak{l}_X \sharp^B F - F \sharp^B \mathfrak{l}_X, \ \forall F \in \mathcal{S}'(\Xi).$$

A Beals type Criterion

Theorem [I.M.P. Comm.PDE 10]

- A tempered distribution $F \in \mathcal{S}'(\Xi)$ is a symbol of class $S^m_\rho(\Xi)$ $(0 \leq \rho \leq 1)$ iff for any $(p, q) \in \mathbb{N}^2$
  and for all the families $u_1, \ldots, u_p \in \mathcal{X}$ and $\mu_1, \ldots, \mu_q \in \mathcal{X}'$
  we have that:
  $$\mathfrak{s}^-_{m-q\rho} \sharp^B \left(\mathfrak{ad}^B_{u_1} \cdot \ldots \cdot \mathfrak{ad}^B_{u_p} \mathfrak{ad}^B_{\mu_1} \cdot \ldots \cdot \mathfrak{ad}^B_{\mu_q} [F] \right) \in \mathfrak{C}^B(\Xi).$$

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$$\mathfrak{s}^-_{m-q\rho} \sharp^B \left(\mathfrak{ad}^B_{u_1} \cdot \ldots \cdot \mathfrak{ad}^B_{u_p} \mathfrak{ad}^B_{\mu_1} \cdot \ldots \cdot \mathfrak{ad}^B_{\mu_q} [F] \right) \in \mathfrak{C}^B(\Xi).$$
- The following two families of seminorms: $\|\mathfrak{s}^-_{m-|\alpha|\rho} \partial^\alpha_\xi \partial^a_x F\|_\infty$, with $(a, \alpha) \in \mathbb{N}^{2n}$, and $\|\mathfrak{s}^-_{m-q\rho} \sharp^B \left(\mathfrak{ad}^B_{u_1} \cdot \ldots \cdot \mathfrak{ad}^B_{u_p} \mathfrak{ad}^B_{\mu_1} \cdot \ldots \cdot \mathfrak{ad}^B_{\mu_q} [F] \right) \|_{\mathfrak{C}^B}$, with $(p, q) \in \mathbb{N}^2$ and vectors from Ξ, define equivalent topologies on $S^m_\rho(\Xi)$.

The Bargmann magnetic representation

The magnetic Bargmann transformation

Definition

We fix some $v \in L^2(\mathcal{X})$ with $\|v\|_{L^2(\mathcal{X})} = 1$;

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for any $u \in L^2(\mathcal{X})$ and for any point $X \in \Xi$ we define

$$\mathfrak{B}_v^B(u)(X) := \left\langle \left(\Lambda^A\right)^{-1} v, W^A(-X) \left(\Lambda^A\right)^{-1} u \right\rangle_{L^2(\mathcal{X})} =: \left\langle v, \widetilde{W}^B(-X) u \right\rangle_{L^2(\mathcal{X})}.$$

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We call the **magnetic Bargmann transformation** of u the function

$$\mathfrak{B}^B_v(u) : \Xi \ni X \mapsto \mathfrak{B}^B_v(u)(X) \in \mathbb{C}.$$

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Let us denote by

$$\mathcal{K}_v^B(\Xi) := \mathfrak{B}_v^B\left[L^2\left(\mathcal{X}; \frac{dX}{(2\pi)^d}\right)\right] \subset L^2\left(\Xi; \frac{dX}{(2\pi)^d}\right) \bigcap BC(\Xi)$$

The inverse transformation

Let us compute now the inverse map of the magnetic Bargmann transformation:

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and after some computations we obtain

$$\widetilde{\mathfrak{B}}_v^B(F) = \widetilde{\mathfrak{Op}}^B \left(\mathcal{F}_\Xi F \right) v \in L^2(\mathcal{X}).$$

The reproducing kernels

Definition

Let $\mathfrak{E}^{B,v}$ be the evaluation map on $\mathcal{K}_v^B(\Xi)$ (subspace of the bounded continuous functions on Ξ):

$$\forall X \in \Xi, \quad \mathfrak{E}_X^{B,v}\left[\mathfrak{B}_v^B(u)\right] := \mathfrak{B}_v^B(u)(X).$$

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- For any $X \in \Xi$ there exist a vector $\mathcal{E}_X^{B,v} \in \mathcal{K}_v^B(\Xi)$ such that

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- After some computation we get that

$$\mathcal{E}_X^{B,v}(Y) = \left\langle \widetilde{W}^B(Y)v, \widetilde{W}^B(X)v \right\rangle_{\mathcal{K}_v^B(\Xi)}.$$

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- After some computation we get that

$$\mathcal{E}^{B,v}_X(Y) = \left\langle \widetilde{W}^B(Y)v, \widetilde{W}^B(X)v\right\rangle_{\mathcal{K}^B_v(\Xi)}.$$

- $\mathcal{E}^{B,v}_X(X) = 1, \qquad \left|\mathcal{E}^{B,v}_X(Y)\right| \leq 1.$

The reproducing kernels

Proposition

For any $F \in \mathcal{K}_v^B(\Xi)$ we have in weak sense:

$$(2\pi)^{-d} \int_\Xi dX \left\langle \mathcal{E}_X^{B,v}, F \right\rangle \mathcal{E}_X^{B,v} = F \quad \text{in } L^2\left(\Xi; \frac{dX}{(2\pi)^d}\right).$$

Thus the following weak-operator integral

$$\mathcal{P}_v^B := (2\pi)^{-d} \int_\Xi dX \left|\mathcal{E}_X^{B,v}\right\rangle \left\langle \mathcal{E}_X^{B,v}\right| : L^2\left(\Xi; \frac{dX}{(2\pi)^d}\right) \to \mathcal{K}_v^B(\Xi)$$

is the orthogonal projection on the closed subspace $\mathcal{K}_v^B(\Xi)$.

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$$\mathfrak{Ba}_v^B(\phi) := \mathfrak{B}_v^B \circ \mathfrak{Op}^A(\phi) \circ \widetilde{\mathfrak{B}}_v^B.$$

Finally one obtains

$$\left[\mathfrak{Ba}_v^B(\phi)F\right](X) = \left\langle \widetilde{\mathfrak{B}}_v^B \left[\mathcal{E}_X^{B,v}\right], \mathfrak{Op}^A(\phi)\widetilde{\mathfrak{B}}_v^B(F)\right\rangle.$$

Thank you for your attention !