SNSB **HW #1**
Topics in Galois and Cogalois Theory **Spring 2011**
 February 24, 2011

1. Prove that for any field F of characteristic $p > 0$, the field $F(X)$ of rational fractions in the indeterminate X is not perfect.

2. Show that for any field F of characteristic $p > 0$, the extension $F(X)/F(X^p)$ is normal but not separable, and determine $\mathrm{Gal}\,(F(X)/F(X^p))$.

3. Let F be any field of characteristic $p > 0$, let $n \in \mathbb{N}^*$, let $X_1, \ldots, X_n$ be n indeterminates, and set $K := F(X_1^p, \ldots, X_n^p)$, $L := F(X_1, \ldots, X_n)$ Prove the following statements.
   (a) The field extension L/K is normal but not separable.
   (b) $[L : K] = p^n$.
   (c) If $n \geqslant 2$, then the finite extension L/K has no primitive element.

4. Prove that a field extension E/F is algebraic $\Longleftrightarrow$ any subring A of E containing F is actually a subfield of E.

5. Show that if E/F is a finite extension of degree a prime number, then the set $\mathbb{I}(E/F)$ of all its intermediate fields consists only of F and E. Examine whether the converse is true.

6. Let K be a field, and let A be an associative unital finite dimensional K-algebra, which is not necessarily commutative. Prove the following assertions.
   (a) If $x \in A$ is not a left (or right) zero-divisor, then x is an invertible element of A.
   (b) A is a skew-field $\Longleftrightarrow$ A is an integral domain, and in this case $\mathrm{Min}(u, K)$ is an irreducible polynomial for any $u \in A$.

7. Let F be a field, let t be a transcendental element over F, and let $E \in \mathbb{I}(F(t)/F)$. If $E \neq F$, then show that t is algebraic over E.

8. Determine whether the real number
$$\frac{2\sqrt{\pi} - 3\sqrt[3]{\pi} + 5\sqrt[5]{\pi}}{7\sqrt[7]{\pi} - 11\sqrt[11]{\pi}}$$
is algebraic or transcendental over $\mathbb{Q}$.

9. Let F be a field, and let t be a transcendental element over F. Show that any $u \in F(t)\backslash F$ is transcendental over F.

10. Determine the splitting field and all the roots in it of the following polynomials:
   (a) $X^7 + 5 \in \mathbb{F}_7[X]$.
   (b) $X^7 + 6X + 5 \in \mathbb{F}_7[X]$.