 May 12, 2011

1. Let $F = \mathbb{Q}(i)$ and $E = F(\sqrt[8]{3})$. Prove that E/F is a generalized 8-Kummer extension, but not a classical Kummer extension. (*Hint*: Show that $\mu_8(E) = \{1, -1, i, -i\}$ and $\sqrt{2} \notin E$.)

2. Show that for any odd prime p, the extension $\mathbb{Q}(\zeta_{p^3})/\mathbb{Q}(\zeta_p)$ is p^2-pure, but it is not a generalized p^2-Kummer extension.

3. Give an example of a Cogalois Abelian extension which is not a classical Kummer extension.

4. Show that any finite classical 2-Kummer extension E/F is a 2-Kummer extension with few roots of unity, but it may happen that $\mu_n(E) \not\subseteq F$ for some $n \in \mathbb{N}^*$, $n \neq 2$.

5. Let $F = \mathbb{Q}(\zeta_3)$ and $E = F(\sqrt[3]{2})$. Show that E/F is a classical Kummer extension which is not a Kummer extension with few roots of unity.

6. Give an example of a Cogalois extension which is not quasi-Kummer, and an example of a quasi-Kummer extension which is not Cogalois.

7. Give an example of a finite classical Kummer extension which is not Cogalois.

8. Give an example of a finite quasi-Kummer extension which is not a classical Kummer extension.

9. Prove that the following assertions are equivalent for an algebraic number field E.

   (1) $E/\mathbb{Q}$ is a Galois G-Cogalois extension for some group G.
   (2) $E/\mathbb{Q}$ is an Abelian G-Cogalois extension for some group G.
   (3) There exist finitely many nonzero rational integers $a_1, \dots, a_r$ such that $E = \mathbb{Q}(\sqrt{a_1}, \dots, \sqrt{a_r})$.
   (4) $E/\mathbb{Q}$ is a classical 2-Kummer extension.

10. Show that the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is G-Cogalois for some group G if and only if $n \in \{1, 2, 3, 4, 6, 8, 12, 24\}$.