

1. Prove that an algebraic extension E/F is finite \iff the set $\{[K : F] \mid K \in \mathbb{I}(E/F), [K : F] < \infty\}$ of natural numbers is bounded. Examine whether the result is true or not when the given extension E/F is not assumed to be algebraic.
2. Let E/F be a field extension, and let $K, L \in \mathbb{I}(E/F)$. Prove the following assertions.
 - (a) If L/F is an algebraic extension, then so is also KL/K .
 - (b) If K/F and L/F are finite extensions, then so is KL/F , and in this case

$$[KL : F] \leq [K : F] \cdot [L : F],$$

with equality if $[K : F]$ and $[L : F]$ are relatively prime.

3. Prove that any field F is algebraically closed in $F(X)$.
4. Prove that $\text{Min}(t, F(t^n)) = X^n - t^n$ for any field F , any $n \in \mathbb{N}^*$, and any transcendental element t over F .
5. Let \mathbb{A} denote the set of all algebraic numbers, i.e., the algebraic closure of \mathbb{Q} in \mathbb{C} . Prove that any field endomorphism of \mathbb{A} is in fact an automorphism of \mathbb{A} . State and prove a generalization of this property to any algebraic extension E/F .
6. Let F be a field of characteristic $p > 0$, and let $a \in F$. Show prove that the so called *Artin-Schreier* polynomial $f = X^p - X - a$ either has a root in F , in which case all its roots are in F , or it is irreducible. In this case, if ξ is a root of f in a splitting field of f , then the extension $F(\xi)/F$ is a cyclic extension of degree p .
7. Prove that the polynomial $X^p - X + a \in \mathbb{Z}[X]$ is irreducible in $\mathbb{Q}[X]$ in case $p \nmid a$. What about the converse, i.e., when $X^p - X + a \in \mathbb{Z}[X]$ is irreducible in $\mathbb{Q}[X]$ then is it true that necessarily $p \nmid a$?
8. Prove that the following statements hold for a finite field extension E/F .
 - (a) $\mathbb{I}(E/F)$ is a finite set \iff the extension E/F is simple.
 - (b) Deduce that any finite separable extension is simple.
 - (c) In case F is an infinite field and $u_1, \dots, u_m \in E$, $m \geq 2$, are separable elements over F , then there exist $c_2, \dots, c_m \in F$ such that $F(u_1, u_2, \dots, u_m) = F(u_1 + c_2u_2 + \dots + c_mu_m)$.
 - (d) Examine whether (c) is true or not when F is finite.
9. For any field F of characteristic $p > 0$, exhibit an infinite number intermediate fields of the finite extension $F(X, Y)/F(X^p, Y^p)$ of degree p^2 .
10. Prove that for any infinite field F , the purely transcendental extension $F(X)/F$ is Galois. Examine whether the result holds for F a finite field.