SNSB **HW #3**

Topics in Galois and Cogalois Theory **Spring 2011**

 March 10, 2011

1. Prove that for any $p \in \mathbb{P}$ and $n \in \mathbb{N}^*$, the polynomial $f = X^p - X + p^n$ is irreducible in $\mathbb{Q}[X]$.

2. Show that a the following two assertions are equivalent for a field F, which is not necessarily commutative.
   (a) F^* is a cyclic group.
   (b) F is finite.

3. For any field F describe $\mathrm{Gal}(F(X)/F)$.

4. Show that the field $\mathbb{Q}$ does not satisfy the condition $C_1(4; -9)$, but satisfies the condition $C_1(4; -2)$.

5. Let F be a field satisfying the condition $C_0(n; a)$. If $\widehat{\widehat{a}}$ denotes the coset of a in the quotient group F^*/F^{*n} and d is the greatest divisor of n such that $a \in F^{*d}$, then prove that

$$\mathrm{ord}(\widehat{\sqrt[n]{a}}) = \mathrm{ord}(\widehat{\widehat{a}}) = n/d.$$

6. Find $[\mathbb{Q}(\sqrt[60]{67500}) : \mathbb{Q}]$.

7. Let F be a field satisfying the condition $C_0(n; a)$, and let $m = [F(\sqrt[n]{a}) : F]$. Show that the map

$$\alpha : \mathbb{D}_m \longrightarrow \mathbb{I}(F(\sqrt[n]{a})/F)), \ \alpha(d) = F(\sqrt[n]{a}^{\,d}),$$

   establishes an anti-isomorphism of lattices.

8. Find all subfields of the field $\mathbb{Q}(\sqrt[n]{a})$, where $n \in \mathbb{N}^*$ and $a \in \mathbb{Q}^*_+$.

9. Let $p \in \mathbb{P}$ and let F be a field such that $\mu_p(F) = \{1\}$. Prove that F satisfies the condition $C_1(p\,; a)$ for any $a \in F^*$.

10. Prove that any finite field $\mathbb{F}_q$ satisfies the condition $C_1(p\,; a)$ for any $a \in \mathbb{F}_q^*$ and any $p \in \mathbb{P}$ with $\gcd(p, q-1) = 1$.