SNSB **HW #5**
Topics in Galois and Cogalois Theory **Spring 2011**
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1. Prove that the following statements hold for a finite extension E/F.
   (a) $|\mathrm{Gal}(E/F)| \leqslant [\,E:F\,]$.
   (b) $|\mathrm{Gal}(E/F)| = [\,E:F\,]$ if and only if the extension E/F is Galois.

2. Let $u = \sqrt[30]{360000} \in \mathbb{R}$, and denote $E = \mathbb{Q}(u)$. Determine
   (a) $\mathrm{Min}(u, \mathbb{Q})$,
   (b) $[E:\mathbb{Q}]$,
   (c) all the subfields of E.

3. Show that $\mathbb{Q}(\sqrt{-3}\,)/\mathbb{Q}$ is a Kneser extension but not a Cogalois extension.

4. Consider the extension $\mathbb{Q}(\sqrt[4]{-9}\,)/\mathbb{Q}$, where $\sqrt[4]{-9}$ is one of the complex roots of the irreducible polynomial $X^4 + 9 \in \mathbb{Q}[X]$, say $\sqrt{6}(1+i)/2$. Prove the following statements.
   (a) $\mathbb{Q}(\sqrt[4]{-9}\,) = \mathbb{Q}(i, \sqrt{6}\,)$.
   (b) If $G = \mathbb{Q}^*\langle \sqrt[4]{-9}\,\rangle$ and $H = \mathbb{Q}^*\langle\, i, \sqrt{6}\,\rangle$, then $|G/\mathbb{Q}^*| = |H/\mathbb{Q}^*| = 4 = [\,\mathbb{Q}(\sqrt[4]{-9}\,) : \mathbb{Q}\,]$.
   (c) The extension $\mathbb{Q}(\sqrt[4]{-9}\,)/\mathbb{Q}$ is simultaneously G-Kneser and H-Kneser, but $G \neq H$.

5. Let $A = \{\,\zeta_3, \sqrt{p_1}, \ldots, \sqrt{p_n}, \ldots\}$, where $p_1, \ldots, p_n, \ldots$ is the sequence of all positive prime numbers. Set $G = \mathbb{Q}^*\langle A\rangle$ and $E = \mathbb{Q}(A) = \mathbb{Q}(G)$. Prove that $|G/\mathbb{Q}^*| = [E:\mathbb{Q}] = \aleph_0$, but the infinite G-radical extension $E/\mathbb{Q}$ is not G-Kneser. This shows that the characterization of finite G-radical extensions E/F being G-Kneser by the equality $|G/F^*| = [E:F]$ fails for infinite G-radical extensions.

6. Let $n \in \mathbb{N}^*$. Prove that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is a $\mathbb{Q}^*\langle\zeta_n\rangle$-Kneser extension if and only if $n = 2^r$ for some $r \in \mathbb{N}$.

7. For what $n \in \mathbb{N}^*$ is $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ a Kneser extension?

8. Let F be an arbitrary field, let $u \in \Omega$ be any root of an irreducible binomial $X^n - a \in F[X]$, and let $m = \mathrm{ord}(\widehat{u})$ in the group Ω^*/F^*, where Ω in an algebraically closed overfield of F. Prove that the extension $F(u)/F$ is $F^*\langle u\rangle$-Kneser.

9. Let $F = \mathbb{F}_2(X^2)$ and $E = \mathbb{F}_2(X)$. Prove the following statements.
   (a) E/F is a quadratic purely inseparable extension.
   (b) E/F is a $T(E/F)$-radical extension.
   (c) The condition (2) from the Kneser Criterion is satisfied.
   (d) The extension E/F is not $T(E/F)$-Kneser.

   This example shows that the separability condition in the Kneser criterion is essential.

10. Prove that a quadratic extension $\mathbb{Q}(\sqrt{d}\,)/\mathbb{Q}$ where $d \neq 1$ is a square-free integer is Cogalois if and only if $d \neq -1, -3$.