 March 31, 2011

1. Exhibit an example of two nonisomorfic fields $E$ and $F$ such that the underlying additive groups $(E,+)$ and $(F,+)$ and the underlying multiplicative groups $(E^*,\cdot)$ and $(F^*,\cdot)$ be isomorphic.

2. Show that a quadratic extension $\mathbb{Q}(\sqrt{d}\,)/\mathbb{Q}$ where $d$ is a square-free integer, is pure if and only if $d \neq -1,\ -3$.

3. Let $F=\mathbb{Q}$, $E_1=\mathbb{Q}(\sqrt{-2}\,)$, and $E_2=\mathbb{Q}(\sqrt{2}\,)$. Show that $E_1/F$ and $E_2/F$ are Cogalois extensions, but their compositum $E_1E_2/F$ is not Cogalois.

4. Let $E/F$ be a finite extension with $|\mathrm{Cog}(E/F)| = [\,E:F\,] = n$.
   (a) Show that $E/F$ is a Cogalois extension whenever $n$ is a prime number, or $n=4$ and $E$ is a subfield of $\mathbb{R}$.
   (b) Is $E/F$ a Cogalois extension for arbitrary $n$?

5. Show that the extension $\mathbb{F}_4/\mathbb{F}_2$ is not Cogalois.

6. Let $n \in \mathbb{N}^*$. Prove that the extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Cogalois if and only if $n=1$ or $n=2$.

7. Prove that the extension $\mathbb{Q}\big(\sqrt{3+\sqrt{2}}\,\big)/\mathbb{Q}$ is neither radical, nor Kneser, nor Cogalois.

8. Prove that $\mathrm{Cog}(\mathbb{Q}\big(\sqrt{3+\sqrt{2}}\,\big)/\mathbb{Q}) = \{\,\widehat{1},\ \widehat{\sqrt{2}}\,\}$.

9. Let $\nu_r = \underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\ldots+\sqrt{2}}}}}_{r\ \text{radicals}}$, and $F_r=\mathbb{Q}(\nu_r)$, $r \geqslant 2$. Show that $F_r/\mathbb{Q}$ is a non Galois extension of degree $2^r$ which is neither radical, nor Kneser, nor Cogalois.

10. Let $d \in \mathbb{Q} \setminus \mathbb{Q}^2$. Show that

$$\mathrm{Cog}\,(\mathbb{Q}(\sqrt{d}\,)/\mathbb{Q}) = \begin{cases} \{\,\widehat{1},\ \widehat{\sqrt{d}}\,\} \cong \mathbb{Z}_2 & \text{if } \ -d,\ -3d \notin \mathbb{Q}^2 \\ \langle \widehat{1+i} \rangle \cong \mathbb{Z}_4 & \text{if } \ -d \in \mathbb{Q}^2 \\ \langle\, \widehat{i\sqrt{3}} \cdot (\widehat{1+i\sqrt{3}}\,)\,\rangle \cong \mathbb{Z}_6 & \text{if } \ -3d \in \mathbb{Q}^2. \end{cases}$$