 April 7, 2011

1. Prove the following assertions.

   (a) $\mathbb{Q}(\sqrt{2+\sqrt{2}}\,) \subseteq \mathbb{Q}(\zeta_{16})$.

   (b) The extension $\mathbb{Q}(\sqrt{2+\sqrt{2}}\,)/\mathbb{Q}$ is not radical, and so, it is not Kneser.

   (c) The extension $\mathbb{Q}(\zeta_{16})/\mathbb{Q}$ is Kneser.

   This shows that a subextension of a Kneser extension is not necessarily Kneser.

2. Prove the following statements.

   (a) For any infinite field F, F is a closed element of the standard Galois connection associated with the extension $F(X)/F$, where X is an indeterminate, but $F(X)/F$ is not an extension with Galois correspondence.

   (b) An infinite normal and separable extension is not necessarily an extension with Galois correspondence.

3. Denote by $\sqrt[4]{-9}$ one of the complex roots, say $\sqrt{6}(1+i)/2$, of the polynomial $X^4+9 \in \mathbb{Q}[X]$. Prove the following statements:

   (a) $\mathbb{Q}(\sqrt[4]{-9}\,)/\mathbb{Q}$ is a $\mathbb{Q}^*\langle \sqrt[4]{-9}\,\rangle$-Kneser extension.

   (b) $K := \mathbb{Q}(\sqrt{6}\,) \in \mathbb{I}(\mathbb{Q}(\sqrt[4]{-9}\,)/\mathbb{Q})$.

   (c) $\mathbb{Q}(\sqrt[4]{-9}\,)/K$ is not a $K^*\mathbb{Q}^*\langle \sqrt[4]{-9}\,\rangle$-Kneser extension, so $\mathbb{Q}(\sqrt[4]{-9}\,)/\mathbb{Q}$ is not a strongly $\mathbb{Q}^*\langle \sqrt[4]{-9}\,\rangle$-Kneser extension.

   (d) For every H with $\mathbb{Q}^* \leqslant H \leqslant \mathbb{Q}^*\langle \sqrt[4]{-9}\,\rangle$, $\mathbb{Q}(\sqrt{6}\,)/\mathbb{Q}$ is not a H-Kneser extension.

   (e) $\mathbb{Q}(\sqrt{6}\,)/\mathbb{Q}$ is a $\mathbb{Q}^*\langle \sqrt{6}\,\rangle$-Kneser extension.

4. Show that the G-radical extension $\mathbb{Q}(\zeta_3)/\mathbb{Q}$, where $G = \mathbb{Q}^*\langle \zeta_3\,\rangle$, is an extension with $G/\mathbb{Q}^*$-Cogalois correspondence which is not G-Cogalois.

5. Prove that the extension $\mathbb{Q}(\zeta_3)/\mathbb{Q}$ is $\mathbb{Q}^*\langle \sqrt{-3}\,\rangle$-Cogalois, but is not Cogalois.

6. Show that any finite separable G-radical extension E/F with $\exp(G/F^*) = 2$ is G-Cogalois.

7. Let E/F be a G-Cogalois extension. Prove that for any $K, L \in \mathbb{I}(E/F)$ with $L \subseteq K$, the extension K/L is $L^*G \cap K^*$-Cogalois.

8. Let E/F be an arbitrary extension such that $\mathrm{Cog}(E/F)$ is a finite group. If $n = \exp(\mathrm{Cog}(E/F))$, then show that the extension E/F is pure if and only if it is n-pure.

9. Give an example of a finite separable radical extension which is not G-Cogalois for any group G.

10. Find $\mathcal{O}_{\mathfrak{S}_n}$ for $n \leqslant 4$, where $\mathfrak{S}_n$ is the symmetric group of degree n.