 April 14, 2011

1. Show that if G is a finite cyclic group of order n, then $\mathcal{O}_G = \mathbb{D}_n$.

2. Prove that the following assertions are equivalent for a finite group G of order n and the canonical map $\omega_G : \mathbb{L}(G) \longrightarrow \mathbb{D}_n,\ H \mapsto |H|$.
   (a) G is a cyclic group.
   (b) The map ω_G is injective.
   (c) The map ω_G is a lattice isomorphism.

3. Prove that an Abelian group G is a group of bounded order if and only if there exists a finite set $\{p_1, \ldots, p_r\}$ of positive prime numbers and a finite family $(G_i)_{1 \leqslant i \leqslant r}$ of Abelian groups, such that
$$G \simeq \bigoplus_{i=1}^{r} G_i \quad \text{and} \quad G_i \simeq \bigoplus_{j \in A_i} \mathbb{Z}_{p_i^{n_{ij}}},$$
where $A_1, \ldots, A_r$ are arbitrary sets, $n_{ij} \in \mathbb{N}^*$, and for each $i = 1, \ldots, r$, the set $\{\, n_{ij} \mid j \in A_i \,\}$ is a bounded subset of $\mathbb{N}^*$.

4. A finite extension E/F is said to have the *unique subfield property*, abbreviated USP, if for every divisor m of $[\,E : F\,]$ there exists a unique intermediate field K of E/F such that $[\,K : F\,] = m$. Show that a finite extension E/F of degree n has the USP if and only if the canonical map
$$\mathbb{I}(E/F) \longrightarrow \mathbb{D}_n,\ K \mapsto [\,K : F\,],$$
is a lattice isomorphism.

5. Prove that the following are equivalent for a finite G-Cogalois extension E/F of degree n.
   (a) E/F has the USP.
   (b) $G/F^* \cong \mathbb{Z}_n$.

6. (*Acosta de Orozco & Vélez* (1982)). Let F be any field, and let $u \in \Omega$ be a root of an irreducible binomial $X^n - a \in F[X]$ with $\gcd(n, e(F)) = 1$, where Ω in an algebraically closed overfield of F. Prove that the extension $F(u)/F$ has the USP if and only if the following two conditions are satisfied.
   (a) $\zeta_p \notin F(u) \setminus F$ for every odd prime divisor p of n.
   (b) If $4 \mid n$, then $\zeta_4 \notin F(u) \setminus F$.

7. With notation and hypotheses of Exercise 6, prove that the following assertions are equivalent.
   (a) The extension $F(u)/F$ has the USP.
   (b) The extension $F(u)/F$ is n-pure.
   (c) The extension $F(u)/F$ is $F^*\langle u \rangle$-Cogalois.
   (d) The extension $F(u)/F$ is G-Cogalois for some group G, and G/F^* is a cyclic group.

8. Prove the following statements.
   (a) Any extension of degree a prime number has the USP.
   (b) A finite G-radical extension having the USP is not necessarily G-Cogalois.
   (c) A finite G-Cogalois extension may fail to have the USP.

9. Show that the condition “G/F^* is a cyclic group” in the statement (d) of Exercise 7 is essential.

10. Prove that if $n \not\equiv 0 \pmod 4$, then the condition “G/F^* is a cyclic group” in the statement (d) of Exercise 7 is superfluous.