SNSB **HW #9**
Topics in Galois and Cogalois Theory **Spring 2011**
 May 5, 2011

1. Let $n \in \mathbb{N}$. Prove that $2\cos(\pi/2^n)$ can be written as a sum of real numbers of type $\pm \sqrt[n_i]{a_i}$, $1 \leqslant i \leqslant r$, where $r, n_1, \ldots, n_r, a_1, \ldots, a_r \in \mathbb{N}^*$, if and only if $n \in \{0, 1, 2\}$.

2. Prove that the following statements are equivalent for an $n \in \mathbb{N}$.

   (a) $\mathbb{Q}(\cos(\pi/2^n))/\mathbb{Q}$ is a Cogalois extension.
   (b) $\mathbb{Q}(\cos(\pi/2^n))/\mathbb{Q}$ is a Kneser extension.
   (c) $\mathbb{Q}(\cos(\pi/2^n))/\mathbb{Q}$ is a radical extension.
   (d) $n \in \{0, 1, 2\}$.

3. For which $n \in \mathbb{N}$ is $\mathbb{Q}(\sin(\pi/2^n))/\mathbb{Q}$ a Cogalois extension?

4. Show that $\mathrm{Cog}\,(\mathbb{Q}(\cos(\pi/2^n))/\mathbb{Q}) = \{\,\widehat{1}, \widehat{\sqrt{2}}\,\}$ for any $n \in \mathbb{N}$, $n \geqslant 2$.

5. Consider the field $E := \mathbb{Q}\,(\sqrt[4]{12}\,, \sqrt[6]{108}\,)$

   (a) Determine $[E : \mathbb{Q}]$.
   (b) Exhibit a vector space basis of the extension $E/\mathbb{Q}$.
   (c) Determine all subfields of E.

6. Show that $[\,\mathbb{Q}(\sqrt[4]{12}, \sqrt[4]{3}\,) : \mathbb{Q}\,] = 8$, and exhibit a vector space basis of the extension $\mathbb{Q}(\sqrt[4]{12}, \sqrt[4]{3}\,)/\mathbb{Q}$.

7. Show that $[\,\mathbb{Q}(\sqrt[6]{18}, \sqrt[6]{162}\,) : \mathbb{Q}\,] = 18$, and exhibit a vector space basis of the extension $\mathbb{Q}(\sqrt[6]{18}, \sqrt[6]{162}\,)/\mathbb{Q}$.

8. Find $[\,\mathbb{Q}(\sqrt{2}, \sqrt[3]{6}, \sqrt[5]{60}\,) : \mathbb{Q}\,]$.

9. Let Ω be an algebraic closure of the field $F := \mathbb{Q}(X)$ of rational fractions in the indeterminate X, and consider the elements

$$x_1 = \sqrt[6]{X^2 + 1} \quad \text{and} \quad x_2 = \sqrt[6]{X^4 - 2X^3 + 2X^2 - 2X + 1}$$

   of Ω. Find $[\,F(x_1, x_2) : F\,]$.

10. Let $r \in \mathbb{N}^*$ and $a_1, \ldots, a_r \in \mathbb{Z}^*$. Prove that

$$[\,\mathbb{Q}\,(\sqrt{a_1}\,, \ldots, \sqrt{a_r}\,) : \mathbb{Q}\,] \;=\; |\,\mathbb{Q}^*\langle\,\sqrt{a_1}\,, \ldots, \sqrt{a_r}\,\rangle/\mathbb{Q}^*\,|.$$